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On the representations of differentials in functional rings and their applications

1. Take the ring $\mathcal{K} := \mathbb{R}\{\{x,t\}\}$, $(x,t) \in \mathbb{R}^2$, of convergent germs of real-valued smooth functions from $C^{(\infty)}(\mathbb{R}^2;\mathbb{R})$ and construct the associated differential polynomial ring $\mathcal{K}\{u\} := \mathcal{K}[\Theta u]$ with respect to a functional variable u, where Θ denotes the standard monoid of all operators generated by commuting differentiations $\partial/\partial x := D_x$ and $\partial/\partial t$. The ideal $I\{u\} \subset \mathcal{K}\{u\}$ is called differential if the condition $I\{u\} = \Theta I\{u\}$ holds.

Consider now the additional differentiation

$$D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}, \tag{1}$$

depending on the functional variable u, which satisfies the Lie-algebraic commutator condition

$$[D_x, D_t] = (D_x u) D_x, \tag{2}$$

for all $(x,t) \in \mathbb{R}^2$. As a simple consequence of (2) the following general (suitably normalized) *representation* of the differentiation (1)

$$D_t = \partial/\partial t + u\partial/\partial x \tag{3}$$

in the differential ring $\mathcal{K}\{u\}$ holds. Impose now on the differentiation (1) a new algebraic constraint

$$D_t^{N-1} u = \bar{z}, \qquad D_t \bar{z} = 0, \tag{4}$$

defining for all natural $N \in \mathbb{N}$ some smooth functional set (or "manifold") $\mathcal{M}^{(N)}$ of functions $u \in \mathbb{R}\{\{x,t\}\}$, and which allows to reduce naturally the initial ring $\mathcal{K}\{u\}$ to the basic ring $\mathcal{K}\{u\}|_{\mathcal{M}_{(N)}} \subseteq \mathbb{R}\{\{x,t\}\}$. In this case the following natural problem of constructing the corresponding representation of differentiation (1) arises: *to find an equivalent linear representation of the reduced differentiation* $D_t|_{\mathcal{M}_{(N)}} : \mathbb{R}^{p(N)}\{\{x,t\}\} \to \mathbb{R}^{p(N)}\{\{x,t\}\}$ *in the functional vector space* $\mathbb{R}^{p(N)}\{\{x,t\}\}$ *for some specially chosen integer dimension* $p(N) \in \mathbb{Z}_+$.

In particular, for an arbitrary $N \in \mathbb{Z}_+$ the following exact matrix expressions

$$l_N[u;\lambda] = \begin{pmatrix} \lambda u_{N-1,x} & u_{N,x} & 0 & \dots & 0 \\ 0 & \lambda u_{N-1,x} & 2u_{N,x} & \ddots & \dots \\ \dots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda u_{N-1,x} & (N-1)u_{N,x} \\ -N\lambda^N & -\lambda^{N-1} N u_{1,x} & \dots & -\lambda^2 N u_{N-2,x} & \lambda(1-N) u_{N-1,x} \end{pmatrix},$$

$$(5)\quad q_N(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & \ddots & \dots & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 \end{pmatrix},$$

polynomial in $\lambda \in \mathbb{C}$, were presented [5, 4, 6] in exact form. Moreover, the same problem is also solvable for the more complicated constraints

$$(6)\qquad D_t^{N-1}u = (D_x\bar{z})^s, \qquad D_t\bar{z} = 0,$$

for arbitrary $s, N \in \mathbb{N}$, equivalent to a generalized Riemann type hydrodynamic flows, and

$$(7)\qquad D_t u - D_x^3 u = 0, \qquad D_x D_t u - u = 0,$$

equivalent to the Lax type integrable nonlinear Korteweg-de Vries and Ostrovsky-Vakhnenko dynamical systems.

2. In the present report we will demonstrate that for $s = 2$, $N = 3$ the problem (6) is completely analytically solvable by means of the differential-algebraic tools, devised in [6]. For the Riemann type hydrodynamical system (6) at $s = 2$ and $N = 2$ it is well known [7] to be a smooth Lax type integrable bi-Hamiltonian flow on the 2π-periodic functional manifold $\bar{M}^2$, whose Lax type representation is given by the following compatible linear system of equations:

$$(8)\quad D_x f = \begin{pmatrix} \bar{z}_x & 0 \\ -\lambda(u + u_x/(2\bar{z}_x) & -\bar{z}_{xx}/(2\bar{z}_x) \end{pmatrix} f, \qquad D_t f = \begin{pmatrix} 0 & 0 \\ -\lambda\bar{z}_x & u_x) \end{pmatrix} f,$$

where $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R}^2)$ and $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter.

Based on the symplectic gradient-holonomic and differential algebraic tools, we will prove the following main proposition.

Proposition 1. *The Riemann type hydrodynamic flow* (6) *at $s = 2$ and $N = 3$ is a bi-Hamiltonian dynamical system on the functional manifold M^3 with respect to two compatible Poissonian structures $\vartheta, \eta : T^*(M^3) \to T(M^3)$*

(9)

$$\vartheta := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2z^{1/2}D_x z^{1/2} \end{pmatrix}, \ \eta := \begin{pmatrix} \partial^{-1} & u_x\partial^{-1} & 0 \\ \partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x & \partial^{-1}z_x - 2z \\ 0 & z_x\partial^{-1} + 2z & 0 \end{pmatrix},$$

possessing an infinite hierarchy of commuting to each other conservation laws and a non-autonomous Lax type representation in the form

$$(10)\ D_x f = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -\lambda z_x & u_x \end{pmatrix} f,$$

$$D_t f = \begin{pmatrix} \lambda^2 u\sqrt{z} & \lambda v\sqrt{z} & z \\ -\lambda^3 tu\sqrt{z} & -\lambda^2 tv\sqrt{z} & -\lambda tz \\ \lambda^4(tuv - u^2) - & -\lambda v_x/\sqrt{z} + & \lambda^2\sqrt{z}(u - tv) - \\ - \lambda^2 u_x/\sqrt{z} & + \lambda^3(tv^2 - uv) & - z_x/2z \end{pmatrix} f,$$

where $\lambda \in \mathbb{R}$ *is an arbitrary spectral parameter and* $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R}^3)$.

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