ADDITIVITY OF INSURANCE PREMIUM

MACIEJ SABLIK

Instytut Matematyki
Uniwersytet Śląski, Katowice

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Part 1.

In the classical book of N. L. Bowers Jr. et al., 1997, see also Gerber, 1979, one can find an explanation of insurance policy in terms of utility functions:
Let X denote a bounded, non-negative random variable (loss to the insurer). EX is *the expected loss* and it is called the *pure* or *net premium*. An insurance company would decide to set the premium for the policy by *loading*, adding to, the pure premium. Consider an insurance company with the initial wealth w and with *a utility function* u to this end.

We assume that the utility function u is an *increasing concave function*. These properties state that the insurer is *a risk averter*. Denote the class of all feasible risks by symbol $\aleph$, the *insurance premium principle* is a real function $H : \aleph \longrightarrow \mathbb{R}$. The $H(X)$ may be computed from the formula

$$u(w) = E\left[u\left(w + H(X) - X\right)\right]. \tag{1}$$

The above method is called *zero utility principle* if $w = 0$. Indeed, any insurance premium principle may be reduced to zero utility principle (cf. S. Heilpern, 2003).

Choquet integral.

Let us call *probability distortion function* any non-decreasing function

$$g : [0, 1] \longrightarrow [0, 1], \tag{2}$$

satisfying boundary conditions

$$g(0) = 0, \; g(1) = 1. \tag{3}$$

For arbitrary random variable X and any $g \in \mathcal{G}$ (the family of all probability distortion functions) we define *Choquet integral* by the formula

$$E_g X := \int_{-\infty}^{0} [g(P(X > t)) - 1] dt + \int_{0}^{\infty} g(P(X > t)) dt, \quad (4)$$

provided both (Riemann) integrals on the left-hand side are finite, they do exist because integrated functions are monotonic.

If X takes finite number of values $x_1 < x_2 < \cdots < x_n$ with probabilities $P(X = x_i) = p_i > 0$, then $E_g X = x_1 + \sum_{i=1}^{n-1} g(q_i)(x_{i+1} - x_i)$, where $q_i = \sum_{k=i+1}^{n} p_k$.

In particular, for $n = 2$, we have

$$E_g X = x_1(1 - g(p_2)) + g(p_2)x_2.$$

Determining the insurance premium.

In the model of *rank-dependent utility* (RDU for short) we get the following equation to determine $H(X)$.

$$u(w) = E_g\left[u\left(w + H(X) - X\right)\right]. \tag{5}$$

Taking $u = \mathrm{id}$ we get

$$H(X) = E_{\overline{g}}(X),$$

where $\overline{g}$ is defined by

$$\overline{g}(x) = 1 - g(1 - x), \ \ x \in [0, 1].$$

In the case where u is exponential, i.e.
$u(x) = \frac{1}{r}\left(1 - \exp(-rx)\right)$, where $r > 0$, we obtain

$$H(X) = \frac{1}{r} \ln E_{\overline{g}} \left(\exp(rX)\right).$$

An interesting problem: when H is additive? More exactly, what are the conditions guaranteeing the following

$$X, Y - \text{ independent} \Longrightarrow H(X+Y) = H(X) + H(Y)? \tag{6}$$

In the paper of S. Heilpern, 2003, we find the following result.

Theorem 1. Let X and Y be independent risks, and let u be either identity or $u(x) = \frac{1}{r}(1 - \exp(-rx))$. Then (6) holds if, and only if, $g = id$.

Proof. A) The case $u = id$, suppose that $X \sim B(1, p)$ and $Y \sim B(1, q)$ are independent risks. Then (6) is equivalent to

$$\overline{g}(p + q - pq) + \overline{g}(pq) = \overline{g}(p) + \overline{g}(q), \tag{7}$$

because $H(X) = \overline{g}(p)$, $H(Y) = \overline{g}(q)$ and $H(X + Y) = \overline{g}(p + q - pq) + \overline{g}(pq)$. This is of course Hosszú equation on the interval $[0, 1]$ and thanks to Lajkó 1974, we get

$$\overline{g}(p) = A(p) + b,$$

for an additive function $A : \mathbb{R} \to \mathbb{R}$ and a constant b. Since g is a probability distortion function, it turns out that

$$g = \mathrm{id}.$$

B) The case $u(x) = \frac{1}{r}(1 - \exp(-rx))$. We get in a similar way the equation

$$\overline{g}(p) + \overline{g}(q) - \overline{g}(p + q - pq) = e^r \overline{g}(pq) - (e^r - 1)\overline{g}(p)\overline{g}(q), \quad (8)$$

which can be rewritten as

$$\frac{\overline{g}(p + q - pq) - \overline{g}(p) - \overline{g}(q) + \overline{g}(pq)}{e^r - 1} = \overline{g}(p)\overline{g}(q) - \overline{g}(pq),$$

or, still more generally,

$$h(p+q-pq)-h(p)-h(q)+h(pq) = \overline{g}(p)\overline{g}(q)-\overline{g}(pq), \quad p, q \in [0, 1]. \quad (9)$$

Heilpern solves (7) and (8) differentiating $\overline{g}$ twice and using other regularity properties. At least (7) can be solved using techniques of the theory of functional equations. As to (8) or (9) we conjectured that actually (9) can be equivalently split into a system

$$\begin{cases} g(xy) = g(x)g(y), \\ h(x+y-xy) + h(xy) = h(x) + h(y), \end{cases} \tag{10}$$

for $x, y \in [0,1]$. (This phenomenon is called *alienation*, see cf. J. Dhombres 1988, R. Ger 1998, 2000, 2010 or R. Ger, L. Reich 2010).

However, during the meeting 51st ISFE, held in Rzeszów, Poland in June, Gy. Maksa yielded a non-trivial example of a function solving (9) but not satisfying (10). Namely, let $M : [0,1] \longrightarrow \mathbb{R}$ be a multiplicative function, i.e.

$$M(xy) = M(x)M(y).$$

(For instance $M(t) = t^p,\ t \in [0,1]$ where $p > 0$ is arbitrary.) Put

$$h(x) = M(1-x),\ x \in [0,1],$$

and

$$g(x) = 1 - M(1-x) = \overline{M}(x),\ x \in [0,1].$$

The pair (g,h) solves (9) but usually (g,h) is not a solution of (10).

On the other hand, one can obtain positive results on alienation of Hosszú and other Cauchy equations. In particular we obtained

Theorem 2. Hosszú equation

$$h(x+y-xy)+h(xy)=h(x)+h(y)$$

and logarithmic equation

$$g(xy)=g(x)+g(y)$$

for functions $g, h : (0,1) \longrightarrow \mathbb{R}$ are alien.

Theorem 3. Hosszú equation

$$h(x+y-xy)+h(xy)=h(x)+h(y)$$

and logarithmic equation

$$g(xy)=g(x)+g(y)$$

for functions $g,h:[0,1]\longrightarrow\mathbb{R}$ are alien.

Theorem 4. Hosszú equation

$$h(x+y-xy)+h(xy)=h(x)+h(y)$$

and additive equation

$$f(x+y)=f(x)+f(y)$$

for functions $f, h : \mathbb{R} \longrightarrow \mathbb{R}$ are alien.

While proving theorems 2 - 4, we apply a method used by R. Ger in 1998. For instance, let us consider the theorem 3. Define the Cauchy difference by $G(x,y) := g(x) + g(y) - g(xy)$, for all $x, y \in (0,1)$. Then G satisfies the cocycle equation:

$$G(xy,z) + G(x,y) = G(x,yz) + G(y,z),$$

for all $x,\ y,\ z \in (0,1)$. But in our case

$$G(x,y) = h(x+y-xy) - h(x) - h(y) + h(xy)$$

whence it easily follows that h satisfies

$$h(xy+z-xyz)+h(x+y-xy) = h(x+yz-xyz)+h(y+z-yz) \tag{11}$$

for all all $x,\ y,\ z \in (0,1)$.

It turns out that (11) is nothing but equation (11) from K. Lajkó's paper (1974). Lajkó proves that actually h solves (11) if and only if it has the form

$$h(x) = \begin{cases} \overline{A}(x - \frac{1}{2}) + h(\frac{1}{2}) \text{ if } x \in (0, 1), \end{cases} \tag{12}$$

where $\overline{A}$ is the so called quasi-extension of a function $A : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}$ given by

$$A(t) = h(t + \frac{1}{2}) - h(\frac{1}{2}),$$

which is additive on the set

$$D := \{(t, s) \in \mathbb{R}^2 : -\frac{1}{2} < t < 0, \frac{1}{2} + \frac{1}{2t - 1} < s < \frac{1}{2} + t\}.$$

Thus we get

$$h(x) = \tilde{A}(x) + c,$$

for $x \in (0, 1)$, where $\tilde{A} : \mathbb{R} \to \mathbb{R}$ is an additive function. We easily check that h satisfies Hosszú functional equation and therefore the pair (g, h) satisfies the system (10). In other words, equations of Hosszú and Cauchy for logarithmic functions are alien to each other.

An alternative proof has been pointed to me by Károly Lajkó during the 51st ISFE meeting. He observed that if a pair (g, h) satisfies

$$g(x)+g(y)-g(xy)=h(x+y-xy)-h(x)-h(y)+h(xy), \quad (13)$$

for all $x, y \in (0, 1)$ then $(f = g + h, h)$ satisfies

$$h(x+y-xy)=f(x)+f(y)-f(xy),\ x, y \in (0,1). \quad (14)$$

Using a theorem from Lajkó 2001, we get

$$h(x)=A(x)+c, \quad (15)$$

where $A : \mathbb{R} \longrightarrow \mathbb{R}$ is an additive function.

Now, from (14) and (15) it turns out that m given by

$$m(x) = f(x) - A(x) - c,\ x \in (0,1),$$

satisfies

$$m(xy) = m(x) + m(y),\ x,y \in (0,1).$$

But obviously $m = g$, and the proof is completed.

Part 2, generalized Choquet integral.

In 1992 A. Tversky and D. Kahneman when describing the mathematical foundations of Cumulative Prospect Theory introduced the concept of a *generalized Choquet integral.* Namely, take two probability distortion functions (from the class $\mathcal{G}$) and define

$$E_{gh}X = E_g X_+ - E_h(-X)_+.$$

If $h(p) = \overline{g}(p) = 1 - g(1-p)$, then $E_{g\overline{g}}X = E_g X$ and we obtain the premium principle introduced by Heilpern.

M. Kałuszka and M. Krzeszowiec, 2012, provide some examples of solutions of the equation

$$u(w) = E_{gh} u\left(w + H(X) - X\right).$$

In particular,

$$H(X) = \varphi^{-1}\left(w + E_{gh} X\right) - w,$$

if $u(x) = cx$ for some $c > 0$. Here $\varphi(t) = t + \int_0^t [h(P(X > s)) - \overline{g}(P(X > s))]$.

On the other hand, if $u(x) = \frac{1-\exp(-cx)}{a}$ then under some additional assumptions on g and h we get

$$H(X) = \frac{1}{c} \ln \phi^{-1} \left(E_h(\exp(cX)) \right),$$

where

$$\phi(t) = t + \int_0^{t \exp(cw)} \left[h(P(\exp(cX) > s)) - \overline{g}(P(\exp(cX) > s)) \right] ds.$$

Denote by $\mathcal{U}$ the class of all functions which vanish at 0, are increasing and continuous, and by $\mathcal{U}_0$ a subclass of $\mathcal{U}$ which consists of functions given by $u(x) = cx$, $u(x) = \frac{1-\exp(-cx)}{a}$ or $u(x) = \frac{\exp(cx)-1}{-a}$ for some $a, c > 0$.

Kałuszka and Krzeszowiec prove the following theorem.

Theorem 5. Let $u \in \mathcal{U}$. If $g(p) = h(p) = p$, $p \in [0,1]$ then H is additive if, and only if, $u \in \mathcal{U}_0$.

Note that the following Lemma holds:

Lemma 1. A symmetrical function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies

$$f(x,y) - f(a,y) - f(x,b) + f(a,b) = 0 \tag{16}$$

for all $a,\ b < 0 < x,\ y$ if and only if for a function $h : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x,y) = h(x) + h(y) \tag{17}$$

for all $x,\ y \neq 0$.

It is enough to observe that taking $X \sim B(s,p)$ and $Y \sim B(z,q)$ to be independent, and denoting $x = H(X)$ and $y = H(Y)$, we easily see that additivity of H is equivalent to

$$\begin{aligned}0 = {} & u(x-s)u(y-z)u(x+y) \\ & - u(x)u(y-z)u(x+y-s) - u(y)u(x-s)u(x+y-z) \\ & + u(x)u(y)u(x+y-s-z).\end{aligned}$$

Define $f(x, y) = \frac{u(x+y)}{u(x)u(y)}$ for $x,\ y \neq 0$. Then f satisfies (16) and hence, in view of Lemma, there exists a function h such that (17) holds.
Thus we get

$$u(x + y) = u(x)F(y) + u(y)F(x) \tag{18}$$

where $F(x) = u(x)h(x)$. (18) is a particular case of the Levi - Civitá equation.

From results of László Székelyhidi from 1991, we derive the following. Either

$$u(x) = a(x)m(x),\ x \in \mathbb{R},$$

or

$$u(x) = \alpha(m_1(x) - m_2(x)),\ x \in \mathbb{R},$$

where $\alpha \in \mathbb{R}$ is a constant, $a : \mathbb{R} \longrightarrow \mathbb{R}$ is an additive function and m, m_1 and m_2 are exponential functions.

If we assume that u is continuous and increasing (and concave!) then we get exactly the members of $\mathcal{U}_0$.

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