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On certain problems concerning the approximation with interpolatory constraints

Let consider the least squares polynomial approximation by orthogonal polynomials for a discrete case in Hilbert space $l^2[-1,1]$:

(1) $$y(x) = \sum_{j=0}^{r} f_j \phi_j(x), \quad x \in [-1,1],\ x_0 = -1,\ x_{N-1} = 1,\ p \le r < N-1,$$

subjected to the constraints:

(2) $$f(x_{\alpha_k}) = y(x_{\alpha_k}) = \sum_{j=0}^{r} f_j \phi_j(x_{\alpha_k}), \quad 0 \le k \le p$$

where

(3) $$0 \le \alpha_0 < \alpha_1 < \ldots < \alpha_{p-1} < \alpha_p \le N-1.$$

The continuous case is defined in an analogous way but the constraint can be located outside the standard interval too. There exists a specific algorithm designed by W. Gautschi [2] based on splitting of the problem to approximation and interpolation:

(4) $$y(x) = \hat{y}(x) + \sigma(x)\tilde{y}(x)$$

where

(5) $$\hat{y}(x) = \sum_{j=0}^{p} \hat{a}_j \hat{\phi}_j(x)$$

is an interpolating polynomial,

(6) $$\tilde{y}(x) = \sum_{j=0}^{r-p-1} \tilde{a}_j \tilde{\phi}_j(x)$$

is an approximating polynomial,

(7) $$\sigma(x) = \prod_{k=0}^{p} (x - x_{\alpha_k}) \equiv \frac{\hat{\phi}_{p+1}(x)}{A_{p+1,p+1}}$$

is the adjusting term. An analogous splitting is proposed by Bakhasi and Iqbal [1].

For interpolation on $p+1$ nodes, and for approximation we have respectively:
Variant 1: We use simply $f(x)$ as function to be approximated.
Variant 2: We define now a new function for unconstrained approximating term:

(8) $$\check{f}(x) = \frac{f(x) - \hat{y}(x)}{\sigma(x)}.$$

Variant 3: We define for unconstrained approximating term

$$\bar{f}(x) = f(x) - \hat{y}(x). \tag{9}$$

We can improve the results obtained from the 3 variants presented above using the modified formula

$$y(x,\varepsilon) = \hat{y}(x) + \varepsilon\sigma(x)\tilde{y}(x) \tag{10}$$

where ε is unknown.

We build the following functional in the Hilbert space $l^2[-1,1]$:

$$J_1(\varepsilon) = \|f - (\hat{y} + \varepsilon\sigma\tilde{y})\|^2_{l^2[-1,1]} = MIN \tag{11}$$

and we then obtain after some manipulations the searched value of the parameter ε:

$$\varepsilon = \frac{(f - \hat{y}, \sigma\tilde{y})_{L^2[-1,1]}}{\|\sigma\tilde{y}\|_{L^2[-1,1]}}\,. \tag{12}$$

If the value of ε is near to 1 then the initial solution is well defined, otherwise it is poor defined.

The algorithm expressed by (1–3) is implemented as program HEL.

References

[1] M. A. Bakhasi, M. Iqbal, *L^2-approximation of real valued functions with interpolatory constraints*, Journal of Computational and Applied Mathematics 70 (1996), 201–205.

[2] W. Gautschi, *Orthogonal Polynomials, Algorithms and Applications*, Springer, Berlin, Heidelberg, New York 2004.