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On application of the Bernstein polynomials to approximation of functions

In this note we shall consider certain modified Bernstein polynomials of differentiable functions of two variables, which have better approximation properties than classical Bernstein polynomials.

Let $r \in N_0 = \{0, 1, 2, \ldots\}$, $I^2 = [0,1] \times [0,1]$, and let $C^r(I^2)$ be the space of real-valued functions f continuous on I^2 with all partial derivatives $f^{(m)}_{x^{m-i}y^i}$, $m \le r$, and the norm is defined by $\|f\| = \max\limits_{(x,y)\in I^2} |f(x,y)|$ ($C^0(I^2) \equiv C(I^2)$). For $f \in C^r(I^2)$ and $n \in N = \{1, 2, \ldots\}$ we introduce the following modified Bernstein polynomials

$$B_{n;r}(f; x, y) := \sum_{j=0}^{n} \sum_{k=0}^{n} p_{n,j}(x) p_{n,k}(y) \sum_{s=0}^{r} \frac{d^s f(\frac{j}{n}, \frac{k}{n})}{s!}, \qquad (x, y) \in I^2, \tag{1}$$

where

$$p_{n,q}(t) := \binom{n}{q} t^q (1-t)^{n-q}$$

and $d^s f(x_0, y_0)$ is the s-th differential of function f at the point (x_0, y_0) and $\Delta x = x - x_0$, $\Delta y = y - y_0$ ($d^0 f(x_0, y_0) = f(x_0, y_0)$).

If $r = 0$, then by (1) we get the classical Bernstein polynomials of f

$$B_n(f; x, y) := \sum_{j=0}^{n} \sum_{k=0}^{n} p_{n,j}(x) p_{n,k}(y)\ f\Big(\frac{j}{n}, \frac{k}{n}\Big), \quad (x, y) \in I^2, \ \ n \in N.$$

The approximation properties of the modified Bernstein polynomials (1) give the following

Theorem. *Let $r \in N_0$ be a fixed number. Then there exists $M(r) = \mathrm{const.} > 0$ such that for every $f \in C^r(I^2)$ and $n \in N$ we have*

$$\|B_{n;r}(f) - f\| \le M(r)\, n^{-r/2} \sum_{s=0}^{r} \omega\Big(f^{(r)}_{x^{r-s}y^s}; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\Big),$$

where $\omega(g)$ is the modulus of continuity of function $g \in C(I^2)$.