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Convergence of Feller semigroups with applications to some stochastic genetic models

We study convergence of semigroups related to a singular-singularly perturbed abstract Cauchy problem (compare [1] and [2]), generalizing a number of recent models of mathematical biology, including the models of gene expression [2, 3, 5] and gene regulation [4]. Particular attention is paid to irregular convergence of these semigroups, i.e. convergence outside of hydrodynamic or regular space, where convergence follows by the Trotter-Kato theorem.

Given $v, w \in \mathbb{R}^M$, $M \in \mathbb{N}$ we define a compact set $J = \{x \in \mathbb{R}^M : v \leq x \leq w\}$. For fixed $N \in \mathbb{N}$ we consider a stochastic process $\{X(t), t \geq 0\}$, which heuristically can be described as follows. $X(t)$ jumps between $N + 1$ copies of J according to a Markov chain-type mechanism. Between jumps, the process moves along the integral curves of ODEs, different on each copy of J. We investigate asymptotic behavior of $X(t)$ when jump intensities are large.

At the time t, let $\mathbf{x}(t)$ denote the position of $X(t)$ in J and let $\gamma(t)$ indicate which copy of J the process moves on. $X(t) = (\mathbf{x}(t), \gamma(t))$ is an example of a piecewise deterministic Markov process of M.H.A. Davis. Consider a sequence $(X_n(t))_{n\geq 0}$ of such processes. Their conditional expected values are given by $\mathbf{f}_n(x, t) := \mathbb{E}_{x_n,\gamma_n}\mathbf{f}_n(\mathbf{x}_n(t), \gamma_n(t))$, where $(x_n, \gamma_n) := (x_n(0), \gamma_n(0))$. If $\mathbf{f}_n(x, t)$ are smooth enough (e.g. are of class C^1), they satisfy the Cauchy problems

$$\frac{\partial \mathbf{f}_n(x,t)}{\partial t} = \mathcal{A}_0\mathbf{f}_n(x,t) + \kappa_n\mathcal{Q}_n\mathbf{f}_n(x,t), \qquad \mathbf{f}_n(x,0) = \theta_n(x), \qquad n \in \mathbb{N}, \tag{1}$$

where for fixed n, t, $\mathbf{f}_n$ belongs to a Cartesian product $\mathbb{B}$ of $N + 1$ copies of $C = C(J)$, the space of real-valued continuous functions on the set J, equipped with the supremum norm. The operator $\mathcal{A}_0$ with domain $\mathcal{D}$ is an infinitesimal generator of a c_0 semigroup of contractions, describing deterministic movement of the processes along integral curves of ODEs. $\mathcal{Q}_n$ is a sequence of bounded multiplication operators in $\mathbb{B}$, whose entries are continuous functions on J. For $x \in J$, each $\mathcal{Q}_n(x)$ is the intensity matrix of a Markov chain, governing jumps of $X_n(t)$. κ_n is a sequence of non-negative constants such that $\kappa_n \to \infty$ for $n \to \infty$, describing intensity of jumps. We prove a theorem about convergence of semigroups related to (1) for $n \to \infty$, assuming that $\mathcal{Q}_n$ tend in operator norm to a limit operator $\mathcal{Q}$. Assuming that $\mathcal{Q}(x)$ has the stationary distribution $\mathbf{p}_0(x)$ and that $\mathbf{p}_0$ are Lipschitz continuous functions of x, we prove that the solutions of (1) tend to these of

$$\frac{\partial f(x,t)}{\partial t} = \mathbf{p}_0^\top \mathcal{A}_0 f(x,t), \quad f(x,0) = \mathbf{p}_0^\top \theta(x), \quad f \in C^1. \tag{2}$$

This result can be applied to derivation of deterministic approximations of the stochastic Kepler-Elston model of gene regulation ([4]), describing binding of regulatory proteins to regulatory sequence in the gene. Similar deterministic approximation of a stochastic mechanism is used in the Lipniacki model of gene expression ([2, 3, 5]), where random activation or inactivation of a gene stimulates production of mRNA and proteins.

References

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