JOINT MATHEMATICS MEETINGS
AMERICAN MATHEMATICAL SOCIETY
MATHEMATICAL ASSOCIATION OF AMERICA

2017 JOINT MATHEMATICS MEETINGS
Largest Mathematics Meeting in the World
JAN 4-7 (WED - SAT), 2017 | HYATT REGENCY ATLANTA AND MARRIOTT ATLANTA MARQUIS

NONCOMMUTATIVE BORSUK-ULAM-TYPE CONJECTURES REVISITED

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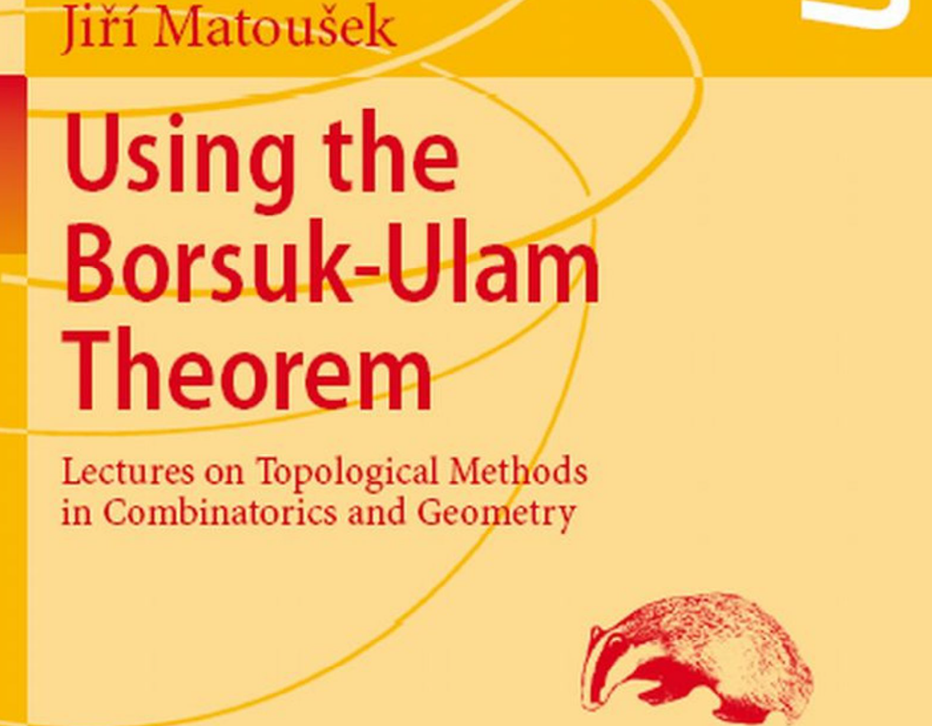
Joint work with Ludwik Dąbrowski and Sergey Neshveyev

Atlanta, 4 January 2017

From 8 to 80


Jiří Matoušek
Using the Borsuk-Ulam Theorem
Lectures on Topological Methods in Combinatorics and Geometry

The Borsuk-Ulam theorem

Theorem (Borsuk-Ulam)

Let n *be a positive natural number. If* $f\colon S^n \to \mathbb{R}^n$ *is continuous, then there exists a pair* $(p, -p)$ *of antipodal points on* S^n *such that* $f(p) = f(-p)$*.*

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Theorem (equivariant formulation)

Let n be a positive natural number. There does not exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^n \to S^{n-1}$.

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Let n be a positive natural number. There does not exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^n \to S^{n-1}$.

Theorem (join formulation)

Let n be a positive natural number. There does not exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^{n-1} * \mathbb{Z}/2\mathbb{Z} \to S^{n-1}$.

Classical generalization

A classical Borsuk-Ulam-type conjecture (Baum, Dąbrowski, H.)

Let X be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group G. Then, for the diagonal action of G on $X * G$, there does not exist a G-equivariant continuous map $f : X * G \to X$.

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Corollary

There does not exist a G-equivariant continuous map $f : X * G \to G$.

For $X = G$ this means that G is not contractible.

The Hilbert-Smith conjecture

Hilbert's fifth problem (Yamabe)

If a connected locally compact Hausdorff topological group G is a projective limit of a sequence of Lie groups, and if there is an open neighbourhood U of the neutral element e containing no subgroup bigger than $\{e\}$, then G is a Lie group.

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The classical Borsuk-Ulam-type theorem implies:

A weak Hilbert-Smith conjecture (Dąbrowski, Chirvasitu, Tobolski)

If G is a compact Hausdorff topological group acting freely and continuously on a topological manifold M, so that the orbit space M/G is finite dimensional, then G is a Lie group.

Associated-vector-bundle theorem

Theorem

Let G be a compact connected semisimple Lie group. Then, there exists a finite-dimensional representation V of G such that for any compact Hausdorff space X equipped with a free G-action, the associated vector bundle

$$(X * G) \overset{G}{\times} V$$

is not stably trivial.

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Note that this theorem implies the second corollary. To prove the theorem, we first show it for $X = G$. The general case follows by pulling back a special-case vector bundle with respect to the map between quotients induced by a G-equivariant map $G * G \to X * G$.

Pointed noncommutative Borsuk-Ulam theorem

Theorem (main)

Let A be a unital C-algebra with a free action $\delta : A \to A \otimes_{\min} H$ of a non-trivial compact quantum group (H, Δ), and let $A \circledast^{\delta} H$ be the equivariant noncommutative join C*-algebra of A and H with the induced free action of (H, Δ). Then, if H admits a character that is not convolution idempotent,*

$$\boxed{\nexists \text{ an } H\text{-equivariant } *\text{-homomorphism } A \longrightarrow A \circledast^{\delta} H}.$$

Furthermore, if A admits a character, then

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This theorem is a straightforward consequence of its special case given by commutative H, and proven by A. Chirvasitu and B. Passer. Now the challange is to remove the red assumption and thus prove the original conjecture of P. F. Baum, L. Dąbrowski and P. M. H.

Noncommutative Brouwer fixed-point theorem

The join of any space with one point is its cone. The cone of a unital C*-algebra A is $\mathcal{C}A := A \circledast \mathbb{C}$. Evaluation at 1 yields a *-homomorphism $\mathrm{ev}_1 : \mathcal{C}A \to A$.

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1. $\nexists$ an H-equivariant $*$-homomorphism $A \to A \circledast^{\delta} H$,
2. $\nexists$ a $*$-homomorphism $\gamma : A \to \mathcal{C}A$ such that $\mathrm{ev}_1 \circ \gamma : A \to A$ is H-colinear.

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Corollary (A noncommutative fixed-point theorem)

If A is a unital C-algebra admitting a free action of a compact quantum group (H, Δ) such that there exists a character on H that is not convolution idempotent, then there does not exist a $*$-homomorphism $\gamma : A \to \mathcal{C}A$ such that $\mathrm{ev}_1 \circ \gamma = id_A$.*

Deformation theorem

Theorem

Let G be a compact connected semisimple Lie group. Let $(C(G_q), \Delta_q)$, $q > 0$, be a family of compact quantum groups that is a q-deformation of $(C(G), \Delta)$. Then, for any $q > 0$ there exists a finite-dimensional left $\mathcal{O}(G_q)$-comodule V_q such that for any unital C-algebra A admitting a character and equipped with a free action of $(C(G_q), \Delta_q)$, the associated finitely generated projective left $(A \circledast_\delta C(G_q))^{\mathrm{co}\, C(G_q)}$-module*

$$\mathcal{P}_{C(G_q)}\big(A \underset{\delta}{\circledast} C(G_q)\big)\Box V_q$$

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As in the classical case, we first prove it for $A = C(G_q)$, and use a character on A to construct an H-equivariant *-homomorphism $A \circledast C(G_q) \to C(G_q) \circledast C(G_q)$. Then we apply the noncommutative pulling-back theorem (H. and Maszczyk).