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UNIVERSITÉ DE FRANCHE-COMTÉ

FROM CLASSICAL TO QUANTUM QUATERNIONIC PROJECTIVE SPACES

Piotr M. Hajac
(IMPAN / University of New Brunswick)

Un travail conjoint avec Tomasz Maszczyk

21 juin 2016

Free actions of compact quantum groups

Let A be a unital C^*-algebra and $\delta : A \to A \otimes_{\min} H$ an injective unital $*$-homomorphism. We call δ a coaction of H on A (or an action of the compact quantum group (H, Δ) on A) if

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H$ (counitality).

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Given a compact quantum group (H, Δ), we denote by $\mathcal{O}(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations.

The Peter-Weyl subalgebra

of A is $\mathcal{P}_H(A) := \{\, a \in A \,|\, \delta(a) \in A \otimes_{\mathrm{alg}} \mathcal{O}(H) \,\}$.

The Peter-Weyl-Galois Theorem

Theorem (P. F. Baum, K. De Commer, P.M.H.)

Let A be a unital C-algebra equipped with an action of a compact quantum group (H, Δ). The following conditions are equivalent:*

1. *The action is free.*
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Put $B = A^{\mathrm{co}H} := \{a \in A \mid \delta(a) = a \otimes 1\}$ (coaction-invariants).

The Peter-Weyl-Galois condition

is the bijectivity of the canonical map
$\mathcal{P}_H(A) \otimes_B \mathcal{P}_H(A) \ni x \otimes y \overset{can}{\longmapsto} (x \otimes 1)\delta(y) \in \mathcal{P}_H(A) \otimes_{\mathrm{alg}} \mathcal{O}(H)$.

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Let V and W be $\mathcal{O}(H)$-comodules (representations of (H, Δ)).

The strong monoidality

is the bijectivity of the natural map
$(\mathcal{P}_H(A) \Box V) \otimes_B (\mathcal{P}_H(A) \Box W) \longrightarrow \mathcal{P}_H(A) \Box (V \otimes_{\mathrm{alg}} W)$.

Main result

Theorem

Let (H,Δ) be a compact quantum group, A and A' (H,Δ)-C-algebras, B and B' the corresponding fixed-point subalgebras, and $f : A \to A'$ an equivariant *-homomorphism. Then, if the action of (H,Δ) on A is free and V is a representation of (H,Δ), the following left B'-modules are isomorphic*

$$B'_f \underset{B}{\otimes} \big(\mathcal{P}_H(A)\Box V\big) \cong \mathcal{P}_H(A')\Box V\,.$$

Here B'_f stands for the B'-B-bimodule with the right action given by f, i.e. $b \cdot c = bf(c)$.

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Corollary

The induced map $(f|_B)_* : K_0(B) \to K_0(B')$ satisfies

$$\boxed{(f|_B)_*\big([\mathcal{P}_H(A)\Box V]\big) = [\mathcal{P}_H(A')\Box V]}\,.$$

18–22 July 2016, the Fields Institute

GEOMETRY, REPRESENTATION THEORY AND THE BAUM-CONNES CONJECTURE

A workshop in honour of Paul F. Baum on the occasion of his 80th birthday organized by Alan Carey, George Elliott, Piotr M. Hajac, and Ryszard Nest.

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- The Fields Institute, University of Toronto, Canada
- The Pennsylvania State University, USA
- National Science Foundation, USA

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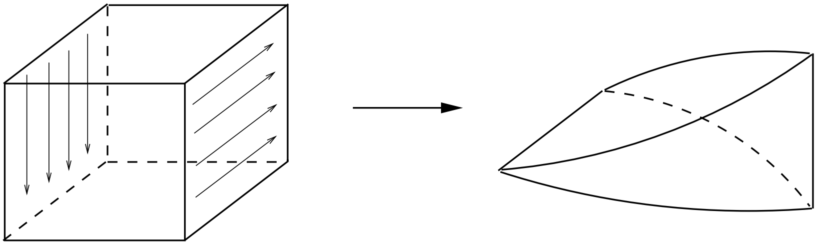
Equivariant join construction

For any topological spaces X and Y, one defines the join space $X * Y$ as the quotient of $[0,1] \times X \times Y$ by a certain equivalence relation:



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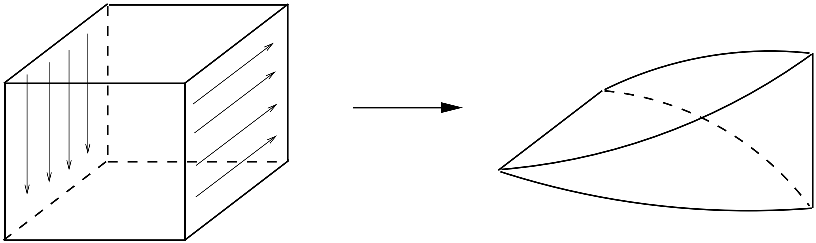

If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G, then the diagonal action of G on the join $X * G$ is again continuous and free.

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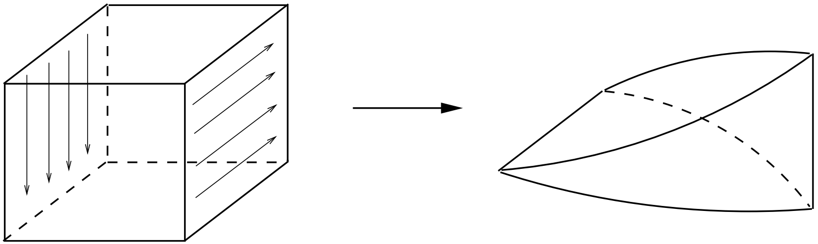

If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G, then the diagonal action of G on the join $X * G$ is again continuous and free. In particular, for the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on S^{n-1}, we obtain a $\mathbb{Z}/2\mathbb{Z}$-equivariant identification $S^n \cong S^{n-1} * \mathbb{Z}/2\mathbb{Z}$ for the antipodal and diagonal actions respectively.

Gauged equivariant join construction

If $Y = G$, we can construct the join G-space $X * Y$ in a different manner: at 0 we collapse $X \times G$ to G as before, and at 1 we collapse $X \times G$ to $(X \times G)/R_D$ instead of X. Here R_D is the equivalence relation generated by

$$\boxed{(x, h) \sim (x', h'),\ \text{where } xh = x'h'}\ .$$

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More precisely, let R'_J be the equivalence relation on $I \times X \times G$ generated by

$$(0,x,h) \sim (0,x',h) \quad \text{and} \quad (1,x,h) \sim (1,x',h'),\ \text{where } xh = x'h'.$$

The formula $[(t,x,h)]k := [(t,x,hk)]$ defines a continuous right G-action on $(I \times X \times G)/R'_J$, and the formula

$$X * G \ni [(t,x,h)] \longmapsto [(t,xh^{-1},h)] \in (I \times X \times G)/R'_J$$

yields a G-equivariant homeomorphism.

Equivariant noncommutative join

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group (H, Δ) acting freely on a unital C*-algebra A, we define its equivariant join with H to be the unital C*-algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0,1], A) \underset{\min}{\otimes} H \;\middle|\; f(0) \in \mathbb{C} \otimes H,\ f(1) \in \delta(A) \right\}.$$

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*The *-homomorphism*

$$\mathrm{id} \otimes \Delta\colon\ C([0,1], A) \underset{\min}{\otimes} H \longrightarrow C([0,1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

defines a free action of the compact quantum group (H, Δ) on the equivariant join C-algebra $A \circledast^{\delta} H$.*

Banach Center
INSTITUTE OF MATHEMATICS PAS

SIMONS FOUNDATION

1 Sep – 30 Nov 2016, Simons Semester in the Banach Center

NONCOMMUTATIVE GEOMETRY THE NEXT GENERATION

Paul F. Baum, Alan Carey, Piotr M. Hajac, Tomasz Maszczyk

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4–17 September, Będlewo & Warsaw, Master Class on:

Noncommutative geometry and quantum groups

1. Cyclic homology
   by Masoud Khalkhali and Ryszard Nest
2. Noncommutative index theory
   by Nigel Higson and Erik Van Erp
3. Topological quantum groups and Hopf algebras
   by Alfons Van Daele and Stanisław L. Woronowicz
4. Structure and classification of C*-algebras
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19 September – 14 October, 20-hour lecture courses:

1. An invitation to C*-algebras by K. R. Strung
2. An introduction to quantum symmetries by R. Ó Buachalla
3. Noncommutative topology for beginners by T. Shulman

Conferences

1. 17–21 Oct. Cyclic homology
   J. Cuntz, P. M. Hajac, T. Maszczyk, R. Nest

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   K. De Commer, P. M. Hajac, R. Ó Buachalla, A. Skalski
4. 21–25 Nov. Structure and classification of C*-algebras
   G. Elliott, K. R. Strung, W. Winter, J. Zacharias

Iterated joins of the quantum $SU(2)$ group

Consider the n-th iteration:

$$SU(2) * \cdots * SU(2) \cong S^{4n+3}.$$

With the diagonal $SU(2)$-action, we obtain

$$S^{4n+3}/SU(2) = \mathbb{HP}^n.$$

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To obtain a q-deformation of this fibration, we take $H := C(SU_q(2))$ and $A := C(S_q^{4n+3})$ equal to the n-times iterated equivariant join of H. We view the fixed-point subalgebra $C(S_q^{4n+3})^{SU_q(2)}$ as the defining C*-algebra $C(\mathbb{HP}_q^n)$ of a *quantum quaternionic projective space*.

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Then we define the noncommutative tautological quaternionic line bundle and its dual as noncommutative complex vector bundles associated through the contragredient representation $V_f^\vee$ of the fundamental represention of $SU_q(2)$ and the fundamental represention V_f itself, respectively.

Quantum quaternionic line bundles

Theorem

For any $n \in \mathbb{N} \setminus \{0\}$ *and* $0 < q \leq 1$, *the noncommutative tautological quaternionic line bundle and its dual are* not *stably trivial as noncommutative complex vector bundles, i.e., the finitely generated projective left* $C(\mathbb{HP}^n_q)$*-modules* $\mathcal{P}_{SU_q(2)}(S_q^{4n+3}) \Box V_f^\vee$ *and* $\mathcal{P}_{SU_q(2)}(S_q^{4n+3}) \Box V_f$ *are not stably free.*

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Proof outline: There exists an $SU_q(2)$-equivariant *-homomorphism $C(S_q^{4n+3}) \to C(SU_q(2)) \circledast^{\Delta} C(SU_q(2)) =: C(S_q^7)$. Hence, by the main theorem, it suffices to prove that the left $C(\mathbb{HP}_q^1)$-modules $\mathcal{P}_{SU_q(2)}(S_q^7)\square V_f^\vee$ and $\mathcal{P}_{SU_q(2)}(S_q^7)\square V_f$ are not stably free.

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Quantum quaternionic principal bundles

Let (H, Δ) be a compact quantum group acting freely on a unital C*-algebra A. It follows from Hopf-Galois theory that, if there exists an H-equivariant *-homomorphism $H \to A$, then the associated A^{coH}-module $\mathcal{P}_H(A)\Box V$ is *free* for any left $\mathcal{O}(H)$-comodule V.

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Consequently, the quantum principal $SU_q(2)$-bundle $S_q^{4n+3} \to \mathbb{HP}_q^n$ is *not* trivializable:

Corollary

*There does not exist a $C(SU_q(2))$-equivariant *-homomorphism*

$$f\colon C(SU_q(2)) \longrightarrow C(S_q^{4n+3}).$$

Quantum Dynamics, 2016–2019

Research and Innovation Staff Exchange network of:
IMPAN (Poland), University of Warsaw (Poland), University of Łódź (Poland), University of Glasgow (G. Britain), University of Aberdeen (G. Britain), University of Copenhagen (Denmark), University of Münster (Germany), Free University of Brussels (Belgium), SISSA (Italy), Penn State University (USA), University of Colorado at Boulder (USA), University of Kansas at Lawrence (USA), University of California at Berkeley (USA), University of Denver (USA), Fields Institute (Canada), University of New Brunswick at Fredericton (Canada), University of Wollongong (Australia), Australian National University (Australia), University of Otago (New Zealand), University Michoacana de San Nicolás de Hidalgo (Mexico).

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