THE K-THEORY OF HEEGAARD QUANTUM LENS SPACES

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Odd-to-even connecting homomorphism

For any one-surjective pullback diagram of algebras

$$\begin{array}{ccccc} A_1 & \longleftarrow & A & \longrightarrow & A_2 \\ & \overset{\pi^1}{\searrow} & & \overset{\pi^2}{\swarrow} & \\ & & A_{12} & & \end{array},$$

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\end{array},
$$

there exists the following long exact sequence in algebraic K-theory:

$$
\cdots \longrightarrow K_1^{\mathrm{alg}}(A_{12}) \xrightarrow{\partial_{10}^{\mathrm{alg}}} K_0(A) \longrightarrow K_0(A_1 \oplus A_2) \longrightarrow K_0(A_{12})
$$

given by the construction: $GL_\infty(A_{12}) \ni U \longmapsto M \in Proj(A)$,

$$
\begin{array}{ccccc}
A_1^n & \longleftarrow & M & \longrightarrow & A_2^n \\
& \searrow^{(\pi^1,\cdots,\pi^1)} & & \swarrow^{(\pi^2,\cdots,\pi^2)} & \\
& & A_{12}^n \overset{U}{\cong} A_{12}^n & &
\end{array}.
$$

Milnor idempotent

Take an invertible matrix $U \in GL_n(A_{12})$ representing a class in $K_1^{\mathrm{alg}}(A_{12})$. There exist liftings $c, d \in M_n(A_1)$ such that $(\pi^1 \otimes \mathrm{id})(c) = U^{-1}$ and $(\pi^1 \otimes \mathrm{id})(d) = U$. Then

$$\boxed{p_U := \begin{pmatrix} (c(2-dc)d, 1) & (c(2-dc)(1-dc), 0) \\ ((1-dc)d, 0) & ((1-dc)^2, 0) \end{pmatrix} \in M_{2n}(A)}\,.$$

is an idempotent matrix such that $M \cong A^{2n} p_U$.

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is an idempotent matrix such that $M \cong A^{2n} p_U$. The assignment

$$\boxed{\partial_{10}^{\mathrm{alg}} : K_1^{\mathrm{alg}}(A_{12}) \ni [U]_{\mathrm{alg}} \longmapsto [p_U] - [I_n] \in K_0(A)}\,,$$

where I_n is the identity matrix of the same size as the matrix U, defines the odd-to-even connecting homomorphism.

Mayer-Vietoris six-term exact sequence

For any one-surjective pullback diagram of unital C*-algebras

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there exists the following six-term exact sequence in K-theory:

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A_1) \oplus K_0(A_2) & \xrightarrow{\pi^1_* - \pi^2_*} & K_0(A_{12}) \\ \partial_{10} \big\uparrow \text{ Higson's argument} & & & & \big\downarrow \\ K_1(A_{12}) & \xleftarrow[\pi^1_* - \pi^2_*]{} & K_1(A_1) \oplus K_1(A_2) & \longleftarrow & K_1(A). \end{array}$$

Milnor's construction for C*-algebras

1. For any unital C*-algebra A_{12}, there is a functorial surjection $K_1^{\mathsf{alg}}(A_{12}) \ni [U]_{\mathsf{alg}} \mapsto [U] \in K_1(A_{12})$, and $K_0^{\mathsf{alg}}(A_{12}) \cong K_0(A_{12})$.

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2. Take a splitting $s\colon K_1(A_{12}) \to K_1^{\mathsf{alg}}(A_{12})$, and define

$$\boxed{\partial_{10} := \partial_{10}^{\mathrm{alg}} \circ s}\,.$$

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3. Use the homotopy invariance of K-theory to prove that $\partial_{10}^{\mathrm{alg}} \circ (s - s') = 0$ for any splittings s and s'.
4. Use the commutativity of the diagram and the exactness of the top row to conclude the exactness of the bottom row:

$$\begin{array}{ccccccc}
K_1^{\mathsf{alg}}(A_1 \oplus A_2) & \longrightarrow & K_1^{\mathsf{alg}}(A_{12}) & \xrightarrow{\partial_{10}^{\mathsf{alg}}} & K_0(A) & \longrightarrow & K_0(A_1 \oplus A_2) \\
\downarrow & & \downarrow & & \| & & \| \\
K_1(A_1 \oplus A_2) & \longrightarrow & K_1(A_{12}) & \xrightarrow{\partial_{10}} & K_0(A) & \longrightarrow & K_0(A_1 \oplus A_2).
\end{array}$$

Heegaard quantum 3-sphere and lens spaces

Definition (P.F. Baum, P.M.H., R. Matthes, W. Szymański)

For $0 \leq p, q, \theta < 1$, θ irrational, the C*-algebra algebra of the Heegaard quantum sphere $C(S^3_{pq\theta})$ is the universal C*-algebra generated by two elements a and b satisfying the relations

$$ab = e^{i2\pi\theta} ba, \quad ab^* = e^{-i2\pi\theta} b^* a,$$
$$a^*a - paa^* = 1 - p, \quad b^*b - qbb^* = 1 - q,$$
$$(1 - aa^*)(1 - bb^*) = 0.$$

$C(S^3_{pq\theta})$ is a $U(1)$-C*-algebra for the action α given by

$$\alpha_{e^{i\phi}}(a) := e^{i\phi} a, \qquad \alpha_{e^{i\phi}}(b) := e^{i\phi} b.$$

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Definition (P.M.H., R. Matthes, W. Szymański)

We define the C*-algebra $C(L^N_{pq\theta})$ of the N-th Heegaard quantum lens space as the fixed-point subalgebra $C(S^3_{pq\theta})^{\mathbb{Z}/N\mathbb{Z}}$ for the action α restricted to $\mathbb{Z}/N\mathbb{Z}$ by the N-th roots of 1.

Equivariant pullback presentation

Theorem (P.F. Baum, P.M.H., R. Matthes, W. Szymański)

Let $\mathcal{T}$ denote the Toeplitz algebra. The $U(1)$-C-algebra $C(S^3_{pq\theta})$ is isomorphic to the following pullback of $U(1)$-C^*-algebras with the natural $U(1)$-action on the $\mathbb{Z}$-parts:*

$$
\begin{array}{ccccc}
 & & C(S^3_{pq\theta}) & & \\
 & \swarrow & & \searrow & \\
\mathcal{T} \underset{\theta}{\rtimes} \mathbb{Z} & & & & \mathcal{T} \underset{-\theta}{\rtimes} \mathbb{Z} \\
 & \overset{\pi^1}{\searrow} & & \overset{\pi^2}{\swarrow} & \\
 & & C(S^1) \underset{\theta}{\rtimes} \mathbb{Z} & &
\end{array}
$$

$$
z_+ \longmapsto Z_+, \quad u_+ \longmapsto U_+, \qquad z_- \longmapsto Z_+^{-1}, \quad u_- \longmapsto Z_+U_+ \,.
$$

The C*-algebra $C(L^N_{pq\theta})$ is isomorphic to the following pullback of C^*-algebras:

$$
\begin{array}{ccccc}
 & & C(L^N_{pq\theta}) & & \\
 & \swarrow & & \searrow & \\
\mathcal{T} \underset{N\theta}{\rtimes} \mathbb{Z} & & & & \mathcal{T} \underset{-N\theta}{\rtimes} \mathbb{Z} \\
 & \searrow^{\tilde{\pi}^1} & & \swarrow^{\tilde{\pi}^2} & \\
 & & C(S^1) \underset{N\theta}{\rtimes} \mathbb{Z} & &
\end{array}
$$

$$
\widetilde{z}_+ \longmapsto Z, \quad \widetilde{u}_+ \longmapsto U, \quad \widetilde{z}_- \longmapsto Z^{-1}, \quad \widetilde{u}_- \longmapsto e^{iN(N-1)\pi\theta} Z^N U \,.
$$

K-maps

$$K_0(\mathcal{T} \underset{N\theta}{\rtimes} \mathbb{Z}) \cong \mathbb{Z} \ni m \xmapsto{\tilde{\pi}^1_*} (m,0) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_0(C(S^1) \underset{N\theta}{\rtimes} \mathbb{Z}),$$

$$K_0(\mathcal{T} \underset{-N\theta}{\rtimes} \mathbb{Z}) \cong \mathbb{Z} \ni n \xmapsto{\tilde{\pi}^2_*} (n,0) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_0(C(S^1) \underset{N\theta}{\rtimes} \mathbb{Z}),$$

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Inserting these K-theory groups and maps into the Mayer-Vietoris six-term exact sequence yields

$$\begin{array}{ccccc}
K_0(C(L^N_{pq\theta})) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(m,n)\to(m-n,0)} & \mathbb{Z} \oplus \mathbb{Z} \\
\uparrow & & & & \downarrow \\
\mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{(-Nn,m-n)\leftarrow(m,n)} & \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{0} & K_1(C(L^N_{pq\theta})).
\end{array}$$

K-groups

Thus we immediately obtain that $K_1(C(L^N_{pq\theta})) = \mathbb{Z}$. Furthermore, we can simplify the six-term exact sequence to the exact sequence

$$0 \to N\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \to K_0(C(L^N_{pq\theta})) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(m,n) \mapsto m-n} \mathbb{Z} \to 0.$$

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Consequently, the sequence

$$0 \to N\mathbb{Z} \hookrightarrow \mathbb{Z} \xrightarrow{f} K_0(C(L^N_{pq\theta})) \to \mathbb{Z} \to 0$$

is exact. Hence $\mathrm{Im} f = \mathbb{Z}/N\mathbb{Z}$. On the other hand, since $\mathbb{Z}$ is projective, $K_0(C(L^N_{pq\theta})) = \mathrm{Im} f \oplus \mathbb{Z}$.

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Theorem

$K_0(C(L^N_{pq\theta})) = \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}$ and $K_1(C(L^N_{pq\theta})) = \mathbb{Z}$.

Main result

Theorem

Let $L_N := \{x \in C(S^3_{pq\theta}) \mid \alpha_{e^{\frac{2\pi i}{N}}}(x) = e^{\frac{2\pi i}{N}} x\} \subset C(S^3_{pq\theta})$. *Then* L_N *is not stably free as a left* $C(L^N_{pq\theta})$*-module.*

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Proof.

1. Find a strong connection. This proves the freeness of the action of $\mathbb{Z}/N\mathbb{Z}$ on $S^3_{pq\theta}$ and yields a projection representing L_N.

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1. Find a strong connection. This proves the freeness of the action of $\mathbb{Z}/N\mathbb{Z}$ on $S^3_{pq\theta}$ and yields a projection representing L_N.
2. Compute the Milnor idempotent.
3. Identify $[L_N] - [1]$ as a torsion generator.
4. Use the projection homomorphism $C(L^N_{pq\theta}) \to \mathcal{T} \rtimes_{N\theta} \mathbb{Z}$ and the functoriality of K_0 to exclude the possibility of L_N being stably free.

□

Matching the idempotents

- Chern-Galois: The C^*-algebra $C(S^3_{pq\theta})$ is isomorphic as a $U(1)$-C^*-algebra to $C(S^3_{00\theta})$. The latter is generated by isometries s and t with the $U(1)$-action given by $\tilde{\alpha}_{e^{i\phi}}(s) = e^{i\phi}s$, $\tilde{\alpha}_{e^{i\phi}}(t) = e^{i\phi}t$. Restrict this action to $\mathbb{Z}/N\mathbb{Z}$, and denote the unitary generator of $C(\mathbb{Z}/N\mathbb{Z})$ by e_N.

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$$C(\mathbb{Z}/N\mathbb{Z}) \ni e_N \longmapsto s^* \otimes s \in C(S^3_{pq\theta}) \otimes C(S^3_{pq\theta})$$

defines a strong connection. Applying Chern-Galois theory, we conclude that $ss^* \in C(L^3_{pq\theta}(N))$ is an idempotent (in fact, a projection) representing the K_0-class of L_N.

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- Milnor: On the other hand, the torsion-part of $K_0(C(L^N_{pq\theta}))$ is generated by the odd-to-even connecting homomorphism ∂_{10} applied to the K_1-class of the unitary $Z \in C(S^1) \rtimes_{N\theta} \mathbb{Z}$. The Milnor construction yields

$$\begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (1-\tilde{z}_+\tilde{z}_+^*, 0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & (\tilde{z}_+\tilde{z}_+^*, 1) \end{pmatrix}.$$

Deriving the concluding contradiction

Finally, using the pullback description of $C(S^3_{00\theta})$, we note that $ss^* = (\tilde{z}_+\tilde{z}_+^*, 1)$. Thus it is clear that

$$\partial_{10}([Z]) = 2[1] - [L_N] - [1] = [1] - [L_N].$$

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If L_N were stably free, then there would exist $k,\, m \in \mathbb{N}$ such that $L_N \oplus C(L^N_{pq\theta})^k \cong C(L^N_{pq\theta})^m$ as modules. Then the foregoing equation would imply

$$\partial_{10}([Z]) = [1] + k[1] - [L_N \oplus C(L^N_{pq\theta})^k] = (k+1-m)[1].$$

However, since $\partial_{10}([Z]) \neq 0$, we conclude that $k+1-m \neq 0$. Therefore $N(k+1-m) \neq 0$ and $N(k+1-m)[1] = 0$.

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Finally, using the pullback description of $C(S^3_{00\theta})$, we note that $ss^* = (\tilde{z}_+\tilde{z}_+^*, 1)$. Thus it is clear that

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However, since $\partial_{10}([Z]) \neq 0$, we conclude that $k+1-m \neq 0$. Therefore $N(k+1-m) \neq 0$ and $N(k+1-m)[1] = 0$. This contradicts the fact that the projection map $C(L^N_{pq\theta}) \to \mathcal{T} \rtimes_{N\theta} \mathbb{Z}$ takes the identity to the identity inducing the map

$$K_0(C(L^N_{pq\theta})) \ni N(k+1-m)[1] \mapsto N(k+1-m)[1_{\rtimes}] \neq 0 \in K_0(\mathcal{T} \underset{N\theta}{\rtimes} \mathbb{Z}) = \mathbb{Z}.$$

Hence L_N is not stably free. □