University of Colorado at Boulder



THE K-THEORY OF HEEGAARD QUANTUM LENS SPACES

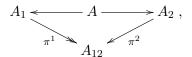
Piotr M. Hajac (IMPAN / University of New Brunswick)

Joint work with Adam Rennie et Bartosz Zieliński

30 March 2016

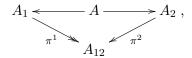
Odd-to-even connecting homomorphism

For any one-surjective pullback diagram of algebras



Odd-to-even connecting homomorphism

For any one-surjective pullback diagram of algebras



there exists the following long exact sequence in algebraic K-theory:

$$\cdots \longrightarrow K_1^{\mathrm{alg}}(A_{12}) \xrightarrow{\partial_{10}^{\mathrm{alg}}} K_0(A) \longrightarrow K_0(A_1 \oplus A_2) \longrightarrow K_0(A_{12})$$

given by the construction: $GL_{\infty}(A_{12}) \ni U \longmapsto M \in Proj(A)$,

$$A_1^n \xrightarrow{\qquad \qquad } A_2^n \ .$$

$$A_{12}^n \cong A_{12}^n \cong A_{12}^n$$

Milnor idempotent

Take an invertible matrix $U \in GL_n(A_{12})$ representing a class in $K_1^{\mathrm{alg}}(A_{12})$. There exist liftings $c,d \in M_n(A_1)$ such that $(\pi^1 \otimes \mathrm{id})(c) = U^{-1}$ and $(\pi^1 \otimes \mathrm{id})(d) = U$. Then

$$p_U := \begin{pmatrix} (c(2-dc)d, 1) & (c(2-dc)(1-dc), 0) \\ ((1-dc)d, 0) & ((1-dc)^2, 0) \end{pmatrix} \in M_{2n}(A).$$

is an idempotent matrix such that $M \cong A^{2n}p_U$.

Milnor idempotent

Take an invertible matrix $U\in GL_n(A_{12})$ representing a class in $K_1^{\mathrm{alg}}(A_{12})$. There exist liftings $c,d\in M_n(A_1)$ such that $(\pi^1\otimes\mathrm{id})(c)=U^{-1}$ and $(\pi^1\otimes\mathrm{id})(d)=U$. Then

$$p_U := \begin{pmatrix} (c(2-dc)d, 1) & (c(2-dc)(1-dc), 0) \\ ((1-dc)d, 0) & ((1-dc)^2, 0) \end{pmatrix} \in M_{2n}(A).$$

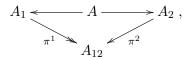
is an idempotent matrix such that $M \cong A^{2n}p_U$. The assignment

$$\partial_{10}^{\mathrm{alg}}: K_1^{\mathrm{alg}}(A_{12}) \ni [U]_{\mathrm{alg}} \longmapsto [p_U] - [I_n] \in K_0(A),$$

where I_n is the identity matrix of the same size as the matrix U, defines the odd-to-even connecting homomorphism.

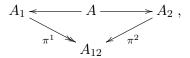
Mayer-Vietoris six-term exact sequence

For any one-surjective pullback diagram of unital C*-algebras



Mayer-Vietoris six-term exact sequence

For any one-surjective pullback diagram of unital C*-algebras



there exists the following six-term exact sequence in K-theory:

• For any unital C*-algebra A_{12} , there is a functorial surjection $K_1^{\mathsf{alg}}(A_{12}) \ni [U]_{\mathsf{alg}} \mapsto [U] \in K_1(A_{12})$, and $K_0^{\mathsf{alg}}(A_{12}) \cong K_0(A_{12})$.

- For any unital C*-algebra A_{12} , there is a functorial surjection $K_1^{\mathsf{alg}}(A_{12}) \ni [U]_{\mathsf{alg}} \mapsto [U] \in K_1(A_{12})$, and $K_0^{\mathsf{alg}}(A_{12}) \cong K_0(A_{12})$.
- 2 Take a splitting $s: K_1(A_{12}) \rightarrow K_1^{\mathsf{alg}}(A_{12})$, and define

$$\boxed{\partial_{10} := \partial_{10}^{\mathrm{alg}} \circ s} \,.$$

- ① For any unital C*-algebra A_{12} , there is a functorial surjection $K_1^{\mathsf{alg}}(A_{12}) \ni [U]_{\mathsf{alg}} \mapsto [U] \in K_1(A_{12})$, and $K_0^{\mathsf{alg}}(A_{12}) \cong K_0(A_{12})$.
- 2 Take a splitting $s: K_1(A_{12}) \rightarrow K_1^{\mathsf{alg}}(A_{12})$, and define

$$\boxed{\partial_{10} := \partial_{10}^{\mathrm{alg}} \circ s}.$$

3 Use the homotopy invariance of K-theory to prove that $\partial_{10}^{\mathrm{alg}} \circ (s-s') = 0$ for any splittings s and s'.

- For any unital C*-algebra A_{12} , there is a functorial surjection $K_1^{\mathsf{alg}}(A_{12}) \ni [U]_{\mathsf{alg}} \mapsto [U] \in K_1(A_{12})$, and $K_0^{\mathsf{alg}}(A_{12}) \cong K_0(A_{12})$.
- 2 Take a splitting $s: K_1(A_{12}) \rightarrow K_1^{\mathsf{alg}}(A_{12})$, and define

$$\partial_{10} := \partial_{10}^{\operatorname{alg}} \circ s \ .$$

- **3** Use the homotopy invariance of K-theory to prove that $\partial_{10}^{\text{alg}} \circ (s s') = 0$ for any splittings s and s'.
- Use the commutativity of the diagram and the exactness of the top row to conclude the exactness of the bottom row:

$$K_1^{\mathsf{alg}}(A_1 \oplus A_2) \longrightarrow K_1^{\mathsf{alg}}(A_{12}) \xrightarrow{\partial_{10}^{\mathsf{alg}}} K_0(A) \longrightarrow K_0(A_1 \oplus A_2)$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \parallel \qquad \qquad \parallel$$

$$K_1(A_1 \oplus A_2) \longrightarrow K_1(A_{12}) \xrightarrow{\partial_{10}} K_0(A) \longrightarrow K_0(A_1 \oplus A_2).$$

Heegaard quantum 3-sphere and lens spaces

Definition (P.F. Baum, P.M.H., R. Matthes, W. Szymański)

For $0 \leq p,q,\theta < 1$, θ irrational, the C*-algebra algebra of the Heegaard quantum sphere $C(S^3_{pq\theta})$ is the universal C*-algebra generated by two elements a and b satisfying the relations

$$ab = e^{i2\pi\theta}ba, \quad ab^* = e^{-i2\pi\theta}b^*a,$$

$$a^*a - paa^* = 1 - p, \quad b^*b - qbb^* = 1 - q,$$

$$(1 - aa^*)(1 - bb^*) = 0.$$

 $C(S^3_{pa heta})$ is a U(1)-C*-algebra for the action α given by

$$\alpha_{e^{i\phi}}(a) := e^{i\phi}a, \qquad \alpha_{e^{i\phi}}(b) := e^{i\phi}b.$$

Heegaard quantum 3-sphere and lens spaces

Definition (P.F. Baum, P.M.H., R. Matthes, W. Szymański)

For $0 \leq p,q,\theta < 1$, θ irrational, the C*-algebra algebra of the Heegaard quantum sphere $C(S^3_{pq\theta})$ is the universal C*-algebra generated by two elements a and b satisfying the relations

$$ab = e^{i2\pi\theta}ba, \quad ab^* = e^{-i2\pi\theta}b^*a,$$

$$a^*a - paa^* = 1 - p, \quad b^*b - qbb^* = 1 - q,$$

$$(1 - aa^*)(1 - bb^*) = 0.$$

 $C(S^3_{pa heta})$ is a U(1)-C*-algebra for the action α given by

$$\alpha_{e^{i\phi}}(a) := e^{i\phi}a, \qquad \alpha_{e^{i\phi}}(b) := e^{i\phi}b.$$

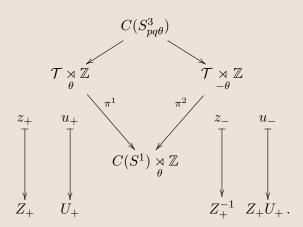
Definition (P.M.H., R. Matthes, W. Szymański)

We define the C*-algebra $C(L_{pq\theta}^N)$ of the N-th Heegaard quantum lens space as the fixed-point subalgebra $C(S_{pq\theta}^3)^{\mathbb{Z}/N\mathbb{Z}}$ for the action α restricted to $\mathbb{Z}/N\mathbb{Z}$ by the N-th roots of 1.

Equivariant pullback presentation

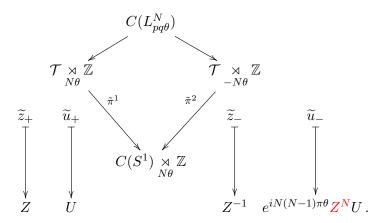
Theorem (P.F. Baum, P.M.H., R. Matthes, W. Szymański)

Let $\mathcal T$ denote the Toeplitz algebra. The U(1)- C^* -algebra $C(S^3_{pq\theta})$ is isomorphic to the following pullback of U(1)- C^* -algebras with the natural U(1)-action on the $\mathbb Z$ -parts:



Analogous equivariant pullback presentation

The C*-algebra $C(L_{pq\theta}^N)$ is isomorphic to the following pullback of C^* -algebras:



K-maps

$$K_{0}(\mathcal{T}\underset{N\theta}{\rtimes}\mathbb{Z}) \cong \mathbb{Z} \ni m \xrightarrow{\tilde{\pi}_{*}^{1}} (m,0) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_{0}(C(S^{1})\underset{N\theta}{\rtimes}\mathbb{Z}),$$

$$K_{0}(\mathcal{T}\underset{-N\theta}{\rtimes}\mathbb{Z}) \cong \mathbb{Z} \ni n \xrightarrow{\tilde{\pi}_{*}^{2}} (n,0) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_{0}(C(S^{1})\underset{N\theta}{\rtimes}\mathbb{Z}),$$

$$K_{1}(\mathcal{T}\underset{N\theta}{\rtimes}\mathbb{Z}) \cong \mathbb{Z} \ni m \xrightarrow{\tilde{\pi}_{*}^{1}} (0,m) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_{1}(C(S^{1})\underset{N\theta}{\rtimes}\mathbb{Z}),$$

$$K_{1}(\mathcal{T}\underset{-N\theta}{\rtimes}\mathbb{Z}) \cong \mathbb{Z} \ni n \xrightarrow{\tilde{\pi}_{*}^{2}} (Nn,n) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_{1}(C(S^{1})\underset{N\theta}{\rtimes}\mathbb{Z}).$$

K-maps

$$K_{0}(\mathcal{T}\underset{N\theta}{\rtimes}\mathbb{Z})\cong\mathbb{Z}\ni m\overset{\tilde{\pi}_{*}^{1}}{\longmapsto}(m,0)\in\mathbb{Z}\oplus\mathbb{Z}\cong K_{0}(C(S^{1})\underset{N\theta}{\rtimes}\mathbb{Z}),$$

$$K_{0}(\mathcal{T}\underset{-N\theta}{\rtimes}\mathbb{Z})\cong\mathbb{Z}\ni n\overset{\tilde{\pi}_{*}^{2}}{\longmapsto}(n,0)\in\mathbb{Z}\oplus\mathbb{Z}\cong K_{0}(C(S^{1})\underset{N\theta}{\rtimes}\mathbb{Z}),$$

$$K_{1}(\mathcal{T}\underset{N\theta}{\rtimes}\mathbb{Z})\cong\mathbb{Z}\ni m\overset{\tilde{\pi}_{*}^{1}}{\longmapsto}(0,m)\in\mathbb{Z}\oplus\mathbb{Z}\cong K_{1}(C(S^{1})\underset{N\theta}{\rtimes}\mathbb{Z}),$$

$$K_{1}(\mathcal{T}\underset{-N\theta}{\rtimes}\mathbb{Z})\cong\mathbb{Z}\ni n\overset{\tilde{\pi}_{*}^{2}}{\longmapsto}(Nn,n)\in\mathbb{Z}\oplus\mathbb{Z}\cong K_{1}(C(S^{1})\underset{N\theta}{\rtimes}\mathbb{Z}).$$

Inserting these K-theory groups and maps into the Mayer-Vietoris six-term exact sequence yields

$$K_0(C(L_{pq\theta}^N)) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(m,n) \to (m-n,0)} \mathbb{Z} \oplus \mathbb{Z}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\mathbb{Z} \oplus \mathbb{Z} \xleftarrow{(-Nn,m-n) \leftarrow (m,n)} \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{0} K_1(C(L_{pq\theta}^N)).$$

K-groups

Thus we immediately obtain that $K_1(C(L_{pq\theta}^N))=\mathbb{Z}$. Furthermore, we can simplify the six-term exact sequence to the exact sequence

$$0 \to N\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \to K_0(C(L_{pq\theta}^N)) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(m,n) \mapsto m-n} \mathbb{Z} \to 0.$$

K-groups

Thus we immediately obtain that $K_1(C(L_{pq\theta}^N))=\mathbb{Z}$. Furthermore, we can simplify the six-term exact sequence to the exact sequence

$$0 \to N\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \to K_0(C(L_{pq\theta}^N)) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(m,n) \mapsto m-n} \mathbb{Z} \to 0.$$

Consequently, the sequence

$$0 \to N\mathbb{Z} \hookrightarrow \mathbb{Z} \xrightarrow{f} K_0(C(L_{pq\theta}^N)) \to \mathbb{Z} \to 0$$

is exact. Hence $\mathrm{Im} f=\mathbb{Z}/N\mathbb{Z}$. On the other hand, since \mathbb{Z} is projective, $K_0(C(L^N_{nq\theta}))=\mathrm{Im} f\oplus \mathbb{Z}$.

K-groups

Thus we immediately obtain that $K_1(C(L_{pq\theta}^N))=\mathbb{Z}$. Furthermore, we can simplify the six-term exact sequence to the exact sequence

$$0 \to N\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \to K_0(C(L_{pq\theta}^N)) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(m,n) \mapsto m-n} \mathbb{Z} \to 0.$$

Consequently, the sequence

$$0 \to N\mathbb{Z} \hookrightarrow \mathbb{Z} \xrightarrow{f} K_0(C(L_{pq\theta}^N)) \to \mathbb{Z} \to 0$$

is exact. Hence $\mathrm{Im} f=\mathbb{Z}/N\mathbb{Z}$. On the other hand, since \mathbb{Z} is projective, $K_0(C(L_{pq\theta}^N))=\mathrm{Im} f\oplus \mathbb{Z}$. Summarising, we have derived:

$\mathsf{Theorem}$

$$K_0(C(L_{pa heta}^N)) = \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}$$
 and $K_1(C(L_{pa heta}^N)) = \mathbb{Z}$.

$\mathsf{Theorem}$

Let $L_N:=\{x\in C(S^3_{pq\theta})\mid \alpha_{e^{\frac{2\pi i}{N}}}(x)=e^{\frac{2\pi i}{N}}x\}\subset C(S^3_{pq\theta}).$ Then L_N is not stably free as a left $C(L^N_{pq\theta})$ -module.

$\mathsf{Theorem}$

Let $L_N:=\{x\in C(S^3_{pq\theta})\mid \alpha_{e^{\frac{2\pi i}{N}}}(x)=e^{\frac{2\pi i}{N}}x\}\subset C(S^3_{pq\theta}).$ Then L_N is not stably free as a left $C(L^N_{pq\theta})$ -module.

Proof.

• Find a strong connection. This proves the freeness of the action of $\mathbb{Z}/N\mathbb{Z}$ on $S^3_{pq\theta}$ and yields a projection representing L_N .

Theorem

Let $L_N:=\{x\in C(S^3_{pq\theta})\mid \alpha_{e^{\frac{2\pi i}{N}}}(x)=e^{\frac{2\pi i}{N}}x\}\subset C(S^3_{pq\theta}).$ Then L_N is not stably free as a left $C(L^N_{pq\theta})$ -module.

Proof.

- Find a strong connection. This proves the freeness of the action of $\mathbb{Z}/N\mathbb{Z}$ on $S^3_{pq\theta}$ and yields a projection representing L_N .
- 2 Compute the Milnor idempotent.

Theorem

Let $L_N:=\{x\in C(S^3_{pq\theta})\mid \alpha_{e^{\frac{2\pi i}{N}}}(x)=e^{\frac{2\pi i}{N}}x\}\subset C(S^3_{pq\theta}).$ Then L_N is not stably free as a left $C(L^N_{pq\theta})$ -module.

Proof.

- **1** Find a strong connection. This proves the freeness of the action of $\mathbb{Z}/N\mathbb{Z}$ on $S^3_{pq\theta}$ and yields a projection representing L_N .
- 2 Compute the Milnor idempotent.
- **3** Identify $[L_N] [1]$ as a torsion generator.

$\mathsf{Theorem}$

Let $L_N:=\{x\in C(S^3_{pq\theta})\mid \alpha_{e^{\frac{2\pi i}{N}}}(x)=e^{\frac{2\pi i}{N}}x\}\subset C(S^3_{pq\theta}).$ Then L_N is not stably free as a left $C(L^N_{pq\theta})$ -module.

Proof.

- **1** Find a strong connection. This proves the freeness of the action of $\mathbb{Z}/N\mathbb{Z}$ on $S^3_{pq\theta}$ and yields a projection representing L_N .
- 2 Compute the Milnor idempotent.
- **3** Identify $[L_N] [1]$ as a torsion generator.
- $\textbf{ 4} \text{ Use the projection homomorphism } C(L_{pq\theta}^N) \!\!\! \to \!\!\! \mathcal{T} \rtimes_{N\theta} \mathbb{Z} \text{ and the functoriality of } K_0 \text{ to exclude the possibility of } L_N \text{ being stably free.}$

Matching the idempotents

• Chern-Galois: The C^* -algebra $C(S^3_{pq\theta})$ is isomorphic as a U(1)- C^* -algebra to $C(S^3_{00\theta})$. The latter is generated by isometries s and t with the U(1)-action given by $\tilde{\alpha}_{e^{i\phi}}(s)=e^{i\phi}s,\ \tilde{\alpha}_{e^{i\phi}}(t)=e^{i\phi}t.$ Restrict this action to $\mathbb{Z}/N\mathbb{Z}$, and denote the unitary generator of $C(\mathbb{Z}/N\mathbb{Z})$ by e_N .

Matching the idempotents

• Chern-Galois: The C^* -algebra $C(S^3_{pq\theta})$ is isomorphic as a U(1)- C^* -algebra to $C(S^3_{00\theta})$. The latter is generated by isometries s and t with the U(1)-action given by $\tilde{\alpha}_{e^{i\phi}}(s)=e^{i\phi}s,\ \tilde{\alpha}_{e^{i\phi}}(t)=e^{i\phi}t.$ Restrict this action to $\mathbb{Z}/N\mathbb{Z}$, and denote the unitary generator of $C(\mathbb{Z}/N\mathbb{Z})$ by e_N . Then the formula

$$C(\mathbb{Z}/N\mathbb{Z}) \ni e_N \longmapsto s^* \otimes s \in C(S^3_{pq\theta}) \otimes C(S^3_{pq\theta})$$

defines a strong connection. Applying Chern-Galois theory, we conclude that $ss^* \in C(L^3_{pq\theta}(N))$ is an idempotent (in fact, a projection) representing the K_0 -class of L_N .

Matching the idempotents

• Chern-Galois: The C^* -algebra $C(S^3_{pq\theta})$ is isomorphic as a U(1)- C^* -algebra to $C(S^3_{00\theta})$. The latter is generated by isometries s and t with the U(1)-action given by $\tilde{\alpha}_{e^{i\phi}}(s)=e^{i\phi}s$, $\tilde{\alpha}_{e^{i\phi}}(t)=e^{i\phi}t$. Restrict this action to $\mathbb{Z}/N\mathbb{Z}$, and denote the unitary generator of $C(\mathbb{Z}/N\mathbb{Z})$ by e_N . Then the formula

$$C(\mathbb{Z}/N\mathbb{Z}) \ni e_N \longmapsto s^* \otimes s \in C(S_{pa\theta}^3) \otimes C(S_{pa\theta}^3)$$

defines a strong connection. Applying Chern-Galois theory, we conclude that $ss^* \in C(L^3_{pq\theta}(N))$ is an idempotent (in fact, a projection) representing the K_0 -class of L_N .

• Milnor: On the other hand, the torsion-part of $K_0(C(L_{pq\theta}^N))$ is generated by the odd-to-even connecting homomorphism ∂_{10} applied to the K_1 -class of the unitary $Z \in C(S^1) \rtimes_{N\theta} \mathbb{Z}$. The Milnor construction yields

$$\left(\begin{array}{cc} (1,1) & (0,0) \\ (0,0) & (1-\tilde{z}_{+}\tilde{z}_{+}^{*},0) \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) - \left(\begin{array}{cc} 0 & 0 \\ 0 & (\tilde{z}_{+}\tilde{z}_{+}^{*},1) \end{array} \right).$$

Deriving the concluding contradiction

Finally, using the pullback description of $C(S^3_{00\theta})$, we note that $ss^*=(\tilde{z}_+\tilde{z}_+^*,1)$. Thus it is clear that

$$\partial_{10}([Z]) = 2[1] - [L_N] - [1] = [1] - [L_N].$$

Deriving the concluding contradiction

Finally, using the pullback description of $C(S^3_{00\theta})$, we note that $ss^*=(\tilde{z}_+\tilde{z}_+^*,1)$. Thus it is clear that

$$\partial_{10}([Z]) = 2[1] - [L_N] - [1] = [1] - [L_N].$$

If L_N were stably free, then there would exist $k,\,m\in\mathbb{N}$ such that $L_N\oplus C(L^N_{pq\theta})^k\cong C(L^N_{pq\theta})^m$ as modules. Then the foregoing equation would imply

$$\partial_{10}([Z]) = [1] + k[1] - [L_N \oplus C(L_{pq\theta}^N)^k] = (k+1-m)[1].$$

However, since $\partial_{10}([Z]) \neq 0$, we conclude that $k+1-m \neq 0$. Therefore $N(k+1-m) \neq 0$ and N(k+1-m)[1] = 0.

Deriving the concluding contradiction

Finally, using the pullback description of $C(S^3_{00\theta})$, we note that $ss^*=(\tilde{z}_+\tilde{z}_+^*,1)$. Thus it is clear that

$$\partial_{10}([Z]) = 2[1] - [L_N] - [1] = [1] - [L_N].$$

If L_N were stably free, then there would exist $k, m \in \mathbb{N}$ such that $L_N \oplus C(L_{pq\theta}^N)^k \cong C(L_{pq\theta}^N)^m$ as modules. Then the foregoing equation would imply

$$\partial_{10}([Z]) = [1] + k[1] - [L_N \oplus C(L_{pq\theta}^N)^k] = (k+1-m)[1].$$

However, since $\partial_{10}([Z]) \neq 0$, we conclude that $k+1-m \neq 0$. Therefore $N(k+1-m) \neq 0$ and N(k+1-m)[1] = 0. This contradicts the fact that the projection map $C(L_{pq\theta}^N) \to \mathcal{T} \rtimes_{N\theta} \mathbb{Z}$ takes the identity to the identity inducing the map

$$K_0(C(L_{pq\theta}^N))\ni N(k+1-m)[1]\mapsto N(k+1-m)[1_{\bowtie}]\neq 0\in K_0(\mathcal{T}_{\bowtie}\mathbb{Z})=\mathbb{Z}.$$

Hence L_N is not stably free. \square