UNIVERSITY of DENVER

PULLING BACK ASSOCIATED NONCOMMUTATIVE VECTOR BUNDLES AND CONSTRUCTING QUANTUM QUATERNIONIC PROJECTIVE SPACES

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31 March 2016

Free actions of compact quantum groups

Let A be a unital C*-algebra and $\delta : A \to A \otimes_{\min} H$ an injective unital $*$-homomorphism. We call δ a coaction of H on A (or an action of the compact quantum group (H, Δ) on A) if

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H$ (counitality).

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Given a compact quantum group (H, Δ), we denote by $\mathcal{O}(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations.

The Peter-Weyl subalgebra

of A is $\mathcal{P}_H(A) := \{\, a \in A \,|\, \delta(a) \in A \otimes_{\mathrm{alg}} \mathcal{O}(H) \,\}$.

The Peter-Weyl-Galois Theorem

Theorem (P. F. Baum, K. De Commer, P.M.H.)

Let A be a unital C-algebra equipped with an action of a compact quantum group (H, Δ). The following conditions are equivalent:*

1. *The action is free.*
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Put $B = A^{\mathrm{co}H} := \{a \in A \mid \delta(a) = a \otimes 1\}$ (coaction-invariants).

The Peter-Weyl-Galois condition

is the bijectivity of the canonical map
$\mathcal{P}_H(A) \otimes_B \mathcal{P}_H(A) \ni x \otimes y \stackrel{can}{\longmapsto} (x \otimes 1)\delta(y) \in \mathcal{P}_H(A) \otimes_{\mathrm{alg}} \mathcal{O}(H)$.

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Let V and W be $\mathcal{O}(H)$-comodules (representations of (H, Δ)).

The strong monoidality

is the bijectivity of the natural map
$(\mathcal{P}_H(A) \Box V) \otimes_B (\mathcal{P}_H(A) \Box W) \longrightarrow \mathcal{P}_H(A) \Box (V \otimes_{\mathrm{alg}} W)$.

Pulling Back Theorem

Theorem

Let (H, Δ) be a compact quantum group, A and A' (H, Δ)-C-algebras, B and B' the corresponding fixed-point subalgebras, and $f : A \to A'$ an equivariant *-homomorphism. Then, if the action of (H, Δ) on A is free and V is a representation of (H, Δ), the following left B'-modules are isomorphic*

$$B'_f \underset{B}{\otimes} \big(\mathcal{P}_H(A) \Box V\big) \cong \mathcal{P}_H(A') \Box V \,.$$

Here B'_f stands for the B'-B-bimodule with the right action given by f, i.e. $b \cdot c = bf(c)$.

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Corollary

The induced map $(f|_B)_*: K_0(B) \to K_0(B')$ satisfies

$$\boxed{(f|_B)_*\big([\mathcal{P}_H(A)\Box V]\big) = [\mathcal{P}_H(A')\Box V]}\,.$$

Strong connections

Let $\mathcal{H}$ be a Hopf algebra with bijective antipode S and $\mathcal{P}$ a right $\mathcal{H}$-comodule algebra for a coaction $\delta : \mathcal{P} \to \mathcal{P} \otimes_{\mathrm{alg}} \mathcal{H}$. We view $\mathcal{H}$ as an $\mathcal{H}$-bicomodule via its comultiplication. We consider $\mathcal{P} \otimes_{\mathrm{alg}} \mathcal{P}$ as an $\mathcal{H}$-bicomodule via

$$\begin{aligned} \mathrm{id} \otimes \delta \qquad & \text{right coaction,} \\ \Big((S^{-1} \otimes \mathrm{id}) \circ \mathrm{flip} \circ \delta\Big) \otimes \mathrm{id} \qquad & \text{left coaction.} \end{aligned}$$

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A strong connection is a unital bicolinear map $\ell : \mathcal{H} \to \mathcal{P} \otimes_{\mathrm{alg}} \mathcal{P}$ such that *multiplication* $\circ\, \ell = \varepsilon$.

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Theorem (T. Brzeziński, P.M.H.)

Let $\mathcal{B}$ be the coaction-invariant subalgebra. The existence of a strong connection is equivalent to the bijectivity of the canonical map $\mathcal{P} \otimes_{\mathcal{B}} \mathcal{P} \to \mathcal{P} \otimes_{\mathrm{alg}} \mathcal{H}$ and the existence of a left $\mathcal{B}$-linear right $\mathcal{H}$-colinear splitting of the multiplication map $\mathcal{B} \otimes \mathcal{P} \to \mathcal{P}$ (equivariant projectivity).

The Chern-Galois proof

Note first that, since $\mathcal{O}(H)$ is cosemisimple, any comodule is a direct sum of finite-dimensional comodules, so that it suffices to prove the theorem for finite-dimesional representations of (H, Δ).

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Furthermore, by the PWG Theorem and the cosemisimplicity of $\mathcal{O}(H)$, there exists a strong connection

$$\ell : \mathcal{O}(H) \longrightarrow \mathcal{P}_H(A) \otimes \mathcal{P}_H(A)$$

on $\mathcal{P}_H(A)$. Next, the equivariance of the *-homomorphism f implies that $\ell' := (f \otimes f) \circ \ell$ is a strong connection on $\mathcal{P}_H(A')$.

The Chern-Galois proof

Now take advantage of Chern-Galois theory to show that applying f componentwise to an idempotent matrix over B representing $\mathcal{P}_H(A)\Box V$ through ℓ is an idempotent matrix over B' of the following block form:

$$e := \begin{pmatrix} e' & 0 \\ r & 0 \end{pmatrix}.$$

Here e' is an idempotent matrix representing $\mathcal{P}_H(A')\Box V$ through ℓ'. It follows from $e^2 = e$ that and $re' = r$.

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Here e' is an idempotent matrix representing $\mathcal{P}_H(A')\Box V$ through ℓ'. It follows from $e^2 = e$ that and $re' = r$. Finally, the computation

$$\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix} \begin{pmatrix} e' & 0 \\ r & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} = \begin{pmatrix} e' & 0 \\ 0 & 0 \end{pmatrix}$$

shows that modules represented respectively by e and e' are isomorphic, i.e. $B'_f \underset{B}{\otimes} \big(\mathcal{P}_H(A)\Box V\big) \cong \mathcal{P}_H(A')\Box V$. $\Box$

The faithful-flatness proof

It follows from the first part of the preceding proof that the canonical map

$$\mathcal{P}_H(A') \underset{B'}{\otimes} \mathcal{P}_H(A') \ni x \otimes y \overset{can'}{\longmapsto} (x \otimes 1)\delta'(y) \in \mathcal{P}_H(A') \otimes \mathcal{O}(H)$$

is bijective, and that $\mathcal{P}_H(A')$ is faithfully flat over B'. Consequently,

$$\widetilde{f} := m_{\mathcal{P}_H(A')} \circ (\mathrm{id} \otimes f) \colon B'_f \underset{B}{\otimes} \mathcal{P}_H(A) \longrightarrow \mathcal{P}_H(A')$$

is an isomorphism if and only if

$$\mathrm{id} \otimes \big(m_{\mathcal{P}_H(A')} \circ (\mathrm{id} \otimes f)\big) \colon \mathcal{P}_H(A') \underset{B'}{\otimes} B'_f \underset{B}{\otimes} \mathcal{P}_H(A) \longrightarrow \mathcal{P}_H(A') \underset{B'}{\otimes} \mathcal{P}_H(A')$$

is an isomorphism. It is the case if and only if

$$m_{\mathcal{P}_H(A')} \otimes f \colon \mathcal{P}_H(A')_f \underset{\mathcal{P}_H(A)}{\otimes} \mathcal{P}_H(A) \underset{B}{\otimes} \mathcal{P}_H(A) \longrightarrow \mathcal{P}_H(A') \underset{B'}{\otimes} \mathcal{P}_H(A')$$

is an isomorphism.

The faithful-flatness proof

Thus, from the commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{P}_H(A')_f \underset{\mathcal{P}_H(A)}{\otimes} \mathcal{P}_H(A) \underset{B}{\otimes} \mathcal{P}_H(A) & \xrightarrow{m_{\mathcal{P}_H(A')} \otimes f} & \mathcal{P}_H(A') \underset{B'}{\otimes} \mathcal{P}_H(A') \\
\Big\downarrow {\scriptstyle (m_{\mathcal{P}_H(A')} \otimes \mathsf{id}) \circ (\mathsf{id} \otimes can)} & & \Big\downarrow {\scriptstyle can'} \\
\mathcal{P}_H(A') \underset{\mathrm{alg}}{\otimes} \mathcal{O}(H) & \xrightarrow{\quad \mathsf{id} \quad} & \mathcal{P}_H(A') \underset{\mathrm{alg}}{\otimes} \mathcal{O}(H)
\end{array}$$

and the bijectivity of the canonical maps, we infer that $\widetilde{f}$ is an isomorphism. Since it is equivariant, we conclude that

$$\widetilde{f} \otimes \mathsf{id} : \left(B'_f \underset{B}{\otimes} \mathcal{P}_H(A) \right) \Box V \longrightarrow \mathcal{P}_H(A') \Box V$$

is an isomorphism of left B'-modules. Finally, as $\mathcal{O}(H)$ is cosemisimple, and any comodule over a cosemisimple Hopf algebra is coflat, it follows that

$$B'_f \underset{B}{\otimes} \left(\mathcal{P}_H(A) \Box V \right) \cong \mathcal{P}_H(A') \Box V \,,$$

as desired. $\square$

Equivariant noncommutative join construction

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group (H, Δ) acting freely on a unital C*-algebra A, we define its equivariant join with H to be the unital C*-algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0,1], A) \underset{\min}{\otimes} H \;\middle|\; f(0) \in \mathbb{C} \otimes H,\ f(1) \in \delta(A) \right\}.$$

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Theorem (P. F. Baum, K. De Commer, P. M. H.)

*The *-homomorphism*

$$\mathrm{id} \otimes \Delta\colon\ C([0,1], A) \underset{\min}{\otimes} H \longrightarrow C([0,1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

defines a free action of the compact quantum group (H, Δ) on the equivariant join C-algebra $A \circledast^{\delta} H$.*

Join Lemma

Lemma (Join Lemma)

Let (H, Δ) *be a compact quantum group,* A *and* A' (H, Δ)*-C*-algebras, and* $F : A \to A'$ *an equivariant *-homomorphism. Then there exists an equivariant *-homomorphism* $f : A \circledast^{\delta} H \to A' \circledast^{\delta'} H$*. Furthermore, if the C*-algebra* H *admits a character* χ*, then one can take as* A *a finitely iterated equivariant join of* H *with itself, and put* $A' = H$*.*

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Proof: Since *-homomorphism F is equivariant, the *-homomorphism

$$\mathrm{id} \otimes F \otimes \mathrm{id} : C([0,1]) \underset{\min}{\otimes} A \underset{\min}{\otimes} H \longrightarrow C([0,1]) \underset{\min}{\otimes} A' \underset{\min}{\otimes} H$$

restricts and corestricts an equivariant *-homomorphism. Next,

$$\mathrm{ev}_{\frac{1}{2}} \otimes \chi \otimes \mathrm{id} : H \circledast^{\Delta} H \longrightarrow H$$

is an equivariant *-homomorphism. By induction, one can extend the domain of this map to an arbitrary finitely iterated equivariant join of H with itself. □

Milnor idempotent

Theorem (P. F. Baum, L. Dąbrowski, P. M. H.)

For any finite-dimensional representation V of a compact quantum group (H, Δ), the associated finitely generated projective module $(H \circledast^{\Delta} H)\Box_H V$ is represented by a Milnor idempotent $p_{U^{-1}}$, where U is a matrix of the representation V.

Quantum quaternionic projective spaces

Consider the defining fibration of the quaternionic projective space:

$$SU(2) * \cdots * SU(2) \cong S^{4n+3}, \quad S^{4n+3}/SU(2) = \mathbb{H}P^n.$$

To obtain a q-deformation of this fibration, we take $H = C(SU_q(2))$ and A equal to a finitely iterated equivariant join of H. The quantum principal $SU_q(2)$-bundle thus given is *not* trivializable:

Theorem

*There does not exist a $C(SU_q(2))$-equivariant *-homomorphism $f\colon C(SU_q(2)) \to A \circledast^{\delta} C(SU_q(2))$ (Borsuk-Ulam).*

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Proof outline: It suffices to show that there exists a finite-dimensional representation V of $SU_q(2)$ for which the associated module is not free. As $C(SU_q(2))$ admits characters, the Join Lemma allows us to apply the Pulling Back Theorem reducing the problem to proving that

$$\big(C(SU_q(2)) \circledast^{\Delta} C(SU_q(2))\big) \Box_{C(SU_q(2))} V$$

is not free. If V is the fundamental representation of $SU_q(2)$, then it follows from index paring considerations applied to the associated Milnor idempotent $p_{U^{-1}}$. □