NONCOMMUTATIVE BORSUK-ULAM-TYPE CONJECTURES REVISITED

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Using the Borsuk-Ulam Theorem

Lectures on Topological Methods in Combinatorics and Geometry

The Borsuk-Ulam Theorem

Theorem (Borsuk-Ulam)

Let n be a positive natural number. If $f\colon S^n \to \mathbb{R}^n$ is continuous, then there exists a pair $(p, -p)$ of antipodal points on S^n such that $f(p) = f(-p)$.

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Theorem (equivariant formulation)

Let n be a positive natural number. There does not exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^n \to S^{n-1}$.

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Theorem (join formulation)

Let n be a positive natural number. There does not exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^{n-1} * \mathbb{Z}/2\mathbb{Z} \to S^{n-1}$.

Classical generalization and corollaries

A classical Borsuk-Ulam-type conjecture

Let X be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group G. Then, for the diagonal action of G on $X * G$, there does not exist a G-equivariant continuous map $f : X * G \to X$.

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Corollary

There does not exist a G-equivariant continuous map $f : X * G \to G$.

For $X = G$ this means that G is not contractible.

Associated-vector-bundle Theorem

Theorem

Let G be a compact connected semisimple Lie group. Then, there exists a finite-dimensional representation V of G such that for any compact Hausdorff space X equipped with a free G-action, the associated vector bundle

$$(X * G) \underset{G}{\times} V$$

is not stably trivial.

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is *not* *stably trivial.*

Note that this theorem implies the second corollary. To prove the theorem, we first show it for $X = G$, take a G-equivariant map $G * G \to X * G$, and apply:

Classical Pulling-Back Theorem

Theorem

Let G be a compact Hausdorff group acting on compact Hausdorff spaces Y and Y', and let $F : Y' \to Y$ be an equivariant continuous map. Then, if the G-action on Y is free and V is a representation of G, the following vector bundles over Y'/G are isomorphic

$$(F|_{Y'/G})^*\big(Y \underset{G}{\times} V\big) \cong Y' \underset{G}{\times} V\,.$$

Corollary

The induced map $(F|_{Y'/G})^* : K^0(Y) \to K^0(Y')$ satisfies

$$\boxed{(F|_{Y'/G})^*\big([Y \underset{G}{\times} V]\big) = [Y' \underset{G}{\times} V]}\,.$$

Compact quantum group

Definition (S. L. Woronowicz)

A compact quantum group is a unital C^*-algebra H with a given unital $*$-homorphism $\Delta\colon H \longrightarrow H \otimes_{\min} H$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\ \Delta\ } & H \underset{\min}{\otimes} H \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \Delta\otimes\mathrm{id}} \\ H \underset{\min}{\otimes} H & \xrightarrow[\mathrm{id}\otimes\Delta]{} & H \underset{\min}{\otimes} H \underset{\min}{\otimes} H \end{array}$$

commutes and the two-sided cancellation property holds:

$$\{(a\otimes 1)\Delta(b) \mid a, b \in H\}^{\mathrm{cls}} = H \underset{\min}{\otimes} H = \{\Delta(a)(1\otimes b) \mid a, b \in H\}^{\mathrm{cls}}.$$

Here "cls" stands for "closed linear span".

Free actions of compact quantum groups

Let A be a unital C*-algebra and $\delta : A \to A \otimes_{\min} H$ an injective unital $*$-homomorphism. We call δ a coaction of H on A (or an action of the compact quantum group (H, Δ) on A) if

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H$ (counitality).

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A coaction δ is called free iff

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Given a compact quantum group (H, Δ), we denote by $\mathcal{O}(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations.

The Peter-Weyl subalgebra

of A is $\mathcal{P}_H(A) := \{\, a \in A \,|\, \delta(a) \in A \otimes_{\mathrm{alg}} \mathcal{O}(H) \,\}$.

Equivariant noncommutative join construction

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group (H, Δ) acting freely on a unital C*-algebra A, we define its equivariant join with H to be the unital C*-algebra

$$A \underset{\delta}{\circledast} H := \left\{ f \in C([0,1], A) \underset{\min}{\otimes} H \;\middle|\; f(0) \in \mathbb{C} \otimes H,\ f(1) \in \delta(A) \right\}.$$

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Theorem (P. F. Baum, K. De Commer, P. M. H.)

*The *-homomorphism*

$$\mathrm{id} \otimes \Delta \colon\ C([0,1], A) \underset{\min}{\otimes} H \longrightarrow C([0,1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

defines a free action of the compact quantum group (H, Δ) on the equivariant join C-algebra $A \circledast_\delta H$.*

Pointed noncommutative Borsuk-Ulam Theorem

Theorem (main)

Let A be a unital C-algebra with a free action $\delta : A \to A \otimes_{\min} H$ of a non-trivial compact quantum group (H, Δ), and let $A \circledast_\delta H$ be the equivariant noncommutative join C*-algebra of A and H with the induced free action of (H, Δ). Then, if H admits a character that is not convolution idempotent,*

$$\boxed{\nexists \text{ an } H\text{-equivariant } *\text{-homomorphism } A \longrightarrow A \circledast_\delta H}.$$

Furthermore, if A admits a character, then

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This theorem is a straightforward consequence of its special case given by commutative H, and proven by A. Chirvasitu and B. Passer. Now the challange is to remove the red assumption and thus prove the original conjecture of P. F. Baum, L. Dąbrowski and P. M. H.

Deformation Theorem

Theorem

Let G be a compact connected semisimple Lie group. Let $(C(G_q), \Delta_q)$, $q > 0$, be a family of compact quantum groups that is a q-deformation of $(C(G), \Delta)$. Then, for any $q > 0$ there exists a finite-dimensional left $\mathcal{O}(G_q)$-comodule V_q such that for any unital C-algebra A admitting a character and equipped with a free action of $(C(G_q), \Delta_q)$, the associated finitely generated projective left $(A \circledast_\delta C(G_q))^{\mathrm{co}\, C(G_q)}$-module*

$$\mathcal{P}_{C(G_q)}\big(A \underset{\delta}{\circledast} C(G_q)\big)\Box V_q$$

is not stably free.

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is not stably free.

As in the classical case, we first prove it for $A = C(G_q)$, use a character on A to construct an H-equivariant *-homomorphism $A \circledast C(G_q) \to C(G_q) \circledast C(G_q)$, and apply:

Noncommutative Pulling-Back Theorem

Theorem (P. M. H. and T. Maszczyk)

Let (H,Δ) be a compact quantum group, C and C' (H,Δ)-C-algebras, B and B' the corresponding fixed-point subalgebras, and $f : C \to C'$ an equivariant *-homomorphism. Then, if the action of (H,Δ) on C is free and V is a representation of (H,Δ), the following left B'-modules are isomorphic*

$$B'_f \underset{B}{\otimes} \big(\mathcal{P}_H(C)\Box V\big) \cong \mathcal{P}_H(C')\Box V\,.$$

Here B'_f stands for the B'-B-bimodule with the right action given by f, i.e. $b\cdot c = bf(c)$.

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Here B'_f stands for the B'-B-bimodule with the right action given by f, i.e. $b \cdot c = bf(c)$.

Corollary

The induced map $(f|_B)_* : K_0(B) \to K_0(B')$ satisfies

$$\boxed{(f|_B)_*\big([\mathcal{P}_H(C)\Box V]\big) = [\mathcal{P}_H(C')\Box V]}\,.$$