The Rokhlin property for compact quantum groups

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Toronto
December 12th 2016

Classification of C^*-algebras

In recent years, the classification programme for simple C^*-algebras has made enormous progress.

Classification builds on certain *regularity properties*, like finite nuclear dimension or $\mathcal{Z}$-stability.

A basic source of examples of (simple) C^*-algebras is the *crossed product construction*.

Under what conditions do regularity properties of a C^*-algebra A pass to a crossed product $G \ltimes A$?

This is where (quantum) groups and the Rokhlin property enter the picture.

Sequence algebras

Let A be a C^*-algebra.

We write

$$\ell^\infty(A) = \ell^\infty(\mathbb{N}, A) = \{(a_n)_{n\in\mathbb{N}} \mid a_n \in A \text{ for all } n \text{ and } \sup_{n\in\mathbb{N}} \|a_n\| < \infty\}$$

$$c_0(A) = c_0(\mathbb{N}, A) = \{(a_n)_{n\in\mathbb{N}} \in \ell^\infty(A) \mid \lim_{n\to\infty} a_n = 0\}.$$

Clearly $c_0(A) \subset \ell^\infty(A)$ is a closed ideal.

Definition

The *sequence algebra* of A is $A_\infty = \ell^\infty(A)/c_0(A)$.

Note that A embeds canonically into A_∞ as equivalence classes of constant sequences.

Definition

The *central sequence algebra* of A is $F_\infty(A) = A_\infty \cap A' \subset A_\infty$.

Actions on sequence algebras

Let G be a (second countable) locally compact group.

Let A be a (separable) C^*-algebra, and let $\mathrm{Aut}(A)$ be the group of all $*$-automorphisms of A.

An *action* of G on A is a group homomorphism $\alpha : G \to \mathrm{Aut}(A), s \mapsto \alpha_s$.

An action α is called *strongly continuous* if for each $a \in A$ the map $G \to A, s \mapsto \alpha_s(a)$ is continuous.

We obtain an induced action $\alpha^\infty : G \to \mathrm{Aut}(\ell^\infty(A))$ by setting

$$\alpha_s^\infty((a_n)) = (\alpha_s(a_n)).$$

This action will typically fail to be strongly continuous.

We also obtain an induced action $\alpha_\infty : G \to \mathrm{Aut}(A_\infty)$, which again will typically not be strongly continuous.

The Rokhlin property - classical case

Let G be a compact group and consider $C(G)$ equipped with the translation action, given by

$$\lambda_t(f)(s) = f(t^{-1}s).$$

Definition (Izumi, Hirshberg-Winter)

Let G be a compact group and let $\alpha : G \to \mathrm{Aut}(A)$ be a strongly continuous action on a unital separable C^*-algebra A. Then α has the Rokhlin property if there exists a unital equivariant $*$-homomorphism $\phi : C(G) \to F_\infty(A)$.

Note that $F_\infty(A) \subset A_\infty$ is preserved under α_∞.

Example

Let $A = C(G)$ with the translation action λ. Then λ has the Rokhlin property.

To go further we need an additional concept...

Sequentially split $*$-homomorphisms

Definition (Barlak-Szabó, 2015)

Let A and B be C^*-algebras. A $*$-homomorphism $\phi : A \to B$ is *sequentially split* if there exists a $*$-homomorphism $\psi : B \to A_\infty$ such that the composition $\psi \circ \phi : A \to A_\infty$ is equal to the canonical inclusion.

Lemma

Let G be a compact group and A a separable C^*-algebra with a strongly continuous action $\alpha : G \to \mathrm{Aut}(A)$. Then the following conditions are equivalent.

a) *α has the Rokhlin property.*

b) *The embedding $A \to C(G) \otimes A, a \mapsto 1 \otimes a$ is G-equivariantly sequentially split.*

Sequentially split $*$-homomorphisms

Theorem (Barlak-Szabó, 2015)

Let $\phi : A \to B$ be a sequentially split $*$-homomorphism. Then the following properties pass from B to A.

- *simplicity.*
- *nuclearity.*
- *finite nuclear dimension.*
- *$\mathcal{Z}$-stability.*
- *real rank zero.*
- *...*

Corollary

If $\alpha : G \to \mathrm{Aut}(A)$ is a Rokhlin action then all of the above properties pass from A to the crossed product $G \ltimes A$ and the fixed point algebra A^G.

Quantum groups

Definition (Woronowicz)

A compact quantum group G is given by a unital C^*-algebra $C(G)$ together with a unital $*$-homomorphism $\Delta : C(G) \to C(G) \otimes C(G)$ satisfying

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

and the density conditions

$$[\Delta(C(G))(C(G) \otimes 1)] = C(G) \otimes C(G) = [\Delta(C(G))(1 \otimes C(G))].$$

Here brackets [ ] denote the closed linear span and all tensor products are minimal.

We shall assume throughout that $C(G) = C^{\mathrm{red}}(G)$ is in reduced form, that is, the Haar state on $C(G)$ is assumed to be faithful.

Actions of quantum groups

Definition

Let G be a compact quantum group and let A a C^*-algebra. A (strongly continuous) action of G on A is a $*$-homomorphism $\alpha : A \to C(G) \otimes A$ satisfying

$$(\Delta \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \alpha) \circ \alpha$$

and the density condition

$$[\alpha(A)(C(G) \otimes 1)] = C(G) \otimes A.$$

We also say that (A, α) is a G-C^*-algebra in this case.

Example

The C^*-algebra $A = C(G)$ equipped with $\alpha = \Delta$ is a G-C^*-algebra.

Some technicalities

If $\alpha : A \to C(G) \otimes A$ is a (strongly continuous) action of a compact quantum group G on a C^*-algebra A then in general we do *not* obtain strongly continuous induced actions on $\ell^\infty(A)$ and A_∞.

However, we obtain an "action"

$$\alpha^\infty : \ell^\infty(A) \to \ell^\infty(C(G) \otimes A), \qquad \alpha^\infty((a_n)) = (\alpha(a_n)).$$

If G is coexact, that is, if $C(G)$ is an exact C^*-algebra, then we also obtain an "action"

$$\alpha_\infty : A_\infty \to (C(G) \otimes A)_\infty, \qquad \alpha_\infty([(a_n)]) = [\alpha^\infty((a_n))].$$

This is sufficient for our purposes.

These "actions" are honest (strongly continuous) actions if the quantum group G is finite, that is, if $C(G)$ is a finite dimensional C^*-algebra.

Braided tensor products

If G is a locally compact group and A, B are G-C^*-algebras, then the diagonal action turns $A \otimes B$ into a G-C^*-algebra.

If G is a quantum group this is no longer true in general.

Instead, if A and B are G-C^*-algebras and in addition A carries a Yetter-Drinfeld structure, there is a *braided tensor product* $A \boxtimes B$ which replaces the ordinary (minimal) tensor product. The braided tensor product is again a G-C^*-algebra such that the action restricts to the given actions on A and B.

We only need a very special instance of this construction:

Lemma

Let G be a compact quantum group and let B be a G-C^*-algebra. Then there is a G-equivariant isomorphism

$$C(G) \boxtimes B \cong C(G) \otimes B$$

where G acts by the translation action on the first tensor factor on the right hand side.

The Rokhlin property - quantum case

Definition

Let G be a compact quantum group G and let $\alpha : A \to C(G) \otimes A$ be an action of G on a separable C^*-algebra A. We say that α has the spatial Rokhlin property if the inclusion map $A \to C(G) \boxtimes A, a \mapsto 1 \boxtimes a$ is G-equivariantly sequentially split.

Equivalently, α has the Rokhlin property if there exists an equivariant $*$-homomorphism $\kappa : C(G) \to A_\infty$ such that

$$a\kappa(f) = \kappa(a_{(-2)} f S(a_{(-1)}))a_{(0)}$$

and $\|\kappa(S(a_{(-1)}))a_{(0)}\| \leq \|a\|$ for all a in (the spectral subalgebra of) A. Here $\alpha(a) = a_{(-1)} \otimes a_{(0)}$.

Example

Let $A = C(G)$ with the translation action $\alpha = \Delta$. Then α has the Rokhlin property.

The Rokhlin property - quantum case

Let $\alpha : A \to C(G) \otimes A$ be an action of a compact quantum group.
The fixed point algebra of α is

$$A^G = \{a \in A \mid \alpha(a) = 1 \otimes a\}.$$

The (reduced) crossed product of α is

$$G \ltimes A = [(C(G) \otimes 1)\alpha(A)] \subset \mathbb{L}(L^2(G) \otimes A).$$

Proposition

Let G be a compact quantum group and let A be a separable C^*-algebra equipped with an action $\alpha : A \to C(G) \otimes A$. If α has the spatial Rokhlin property then

a) *the canonical embedding $A^G \to A$ is sequentially split.*

b) *the canonical embedding $G \ltimes A \to G \ltimes (C(G) \boxtimes A) \cong \mathbb{K}(L^2(G)) \otimes A$ is sequentially split.*

In particular, all the regularity properties mentioned previously pass from A to $G \ltimes A$ and A^G.

Rokhlin property versus freeness

Classically, a Rokhlin action of a compact group on an abelian C^*-algebra of the form $C_0(X)$ induces a free action of G on X. The converse does not hold.

Definition (Ellwood)

If G is a compact quantum group then an action $\alpha : A \to C(G) \otimes A$ on a C^*-algebra A is called free if $[(1 \otimes A)\alpha(A)] = C(G) \otimes A$.

Proposition

Let G be a coexact compact quantum group acting on a separable C^*-algebra A. If the action $\alpha : A \to C(G) \otimes A$ has the Rokhlin property then it is free.

Duality, part I: Approximate representability

Now let G be a discrete quantum group.

If A is a G-C^*-algebra then write $\iota_A : A \to G \ltimes_{\mathsf{red}} A$ for the canonical embedding.

On the crossed product $G \ltimes_{\mathsf{red}} A$ we have a natural *inner action* of G.

Explicitly, this is implemented by the map $\gamma : G \ltimes_{\mathsf{red}} A \to M(C_0(G) \otimes G \ltimes_{\mathsf{red}} A)$ given by

$$\gamma(x) = W^*(1 \otimes x)W$$

where $W \in M(C_0(G) \otimes C^*_{\mathsf{red}}(G))$ is the multiplicative unitary.

Notice that the inclusion map ι_A is G-equivariant.

Definition

Let G be a discrete quantum group, let A be a separable C^*-algebra, and let $\alpha : A \to M(C_0(G) \otimes A)$ be an action of G on A. We say that α is spatially approximately representable if the embedding $\iota_A : A \to G \ltimes_{\mathsf{red}} A$ is G-equivariantly sequentially split.

Duality, part I: Approximate representability

An action $\alpha : A \to M(C_0(G) \otimes A)$ is *representable* if there exists a unitary $V \in M(C_0(G) \otimes A)$ such that $\alpha(a) = V^*(1 \otimes a)V$.

Proposition

Let G be an exact discrete quantum group, let A be a (unital) separable C^*-algebra, and let $\alpha : A \to M(C_0(G) \otimes A)$ be an action of G on A. If α is spatially approximately representable then there exists a unitary representation $V \in M(C_0(G) \otimes A_\infty)$ satisfying

$$(\mathrm{id} \otimes \iota)\alpha(a) = V^*(1 \otimes \iota(a))V$$

for all $a \in A$ and

$$(\mathrm{id} \otimes \alpha_\infty)(V) = V_{23}^* V_{13} V_{23}.$$

Here $\iota : A \to A_\infty$ is the canonical embedding.
Conversely, if G is amenable and if there exists a unitary representation $V \in M(C_0(G) \otimes A_\infty)$ satisfying the above conditions then α is spatially approximately representable.

Duality, part II: Stabilization

If A is a G-C^*-algebra then there exists an induced action on $\mathbb{K}(L^2(G) \otimes A) = \mathbb{K}(L^2(G)) \otimes A$.

Proposition

Let G be a coexact compact quantum group or an exact discrete quantum group. Moreover let $\iota : A \to B$ be a nondegenerate G-equivariant $*$-homomorphism between G-C^*-algebras. Then ι is G-equivariantly sequentially split iff

$$\mathrm{id} \otimes \iota : \mathbb{K}(L^2(G)) \otimes A \to \mathbb{K}(L^2(G)) \otimes B$$

is G-equivariantly sequentially split.

Duality, part III: Duality for Rokhlin actions

By the generalized Takesaki-Takai duality theorem, we have an isomorphism

$$\check{G} \ltimes_{\text{red}} G \ltimes_{\text{red}} A \cong \mathbb{K}(L^2(G))) \otimes A$$

of G-C^*-algebras.

Proposition

Let G be a coexact compact quantum group and let α be an action of G on a separable C^*-algebra A. Then α has the Rokhlin property iff the bidual action $\check{\check{\alpha}}$ of G on $\mathbb{K}(L^2(G)) \otimes A$ has the Rokhlin property. Similarly, let G be an exact discrete quantum group and let α be an action of G on a separable C^*-algebra A. Then α is approximately representable iff the bidual action $\check{\check{\alpha}}$ of G on $\mathbb{K}(L^2(G)) \otimes A$ is approximately representable.

Duality, part III: Duality for Rokhlin actions

Proposition

Let α be an action of a coexact compact quantum group G on a separable C^*-algebra A and let $\check{\alpha}$ be the dual action on $G \ltimes A$. If α has the Rokhlin property then $\check{\alpha}$ is approximately representable.

Proposition

Let G be an exact discrete quantum group and let α be an action of G on a separable G-C^*-algebra A. Denote by $\check{\alpha}$ the dual action on $G \ltimes_{\mathsf{red}} A$. If α is spatially approximately representable then $\check{\alpha}$ has the spatial Rokhlin property.

Duality, part III: Duality for Rokhlin actions

Theorem

Let G be a coexact compact quantum group and let α be an action of G on a separable C^*-algebra A. Then α has the spatial Rokhlin property iff the dual action $\check{\alpha}$ of $\check{G}$ on $G \ltimes A$ is spatially approximately representable. Similarly, let G be an exact discrete quantum group and let α be an action of G on a separable C^*-algebra A. Then α is spatially approximately representable iff the dual action $\check{\alpha}$ of G on $G \ltimes_{\mathsf{red}} A$ has the spatial Rokhlin property.

Examples

Example

Let G be a coamenable compact quantum group acting on $A = C(G)$ by the regular coaction $\alpha = \Delta$. Then α is a Rokhlin action.

Example

Let G be a finite quantum group and set $n = \dim(C(G))$. Then $B = M_n(\mathbb{C})$ is a Yetter-Drinfeld-C^*-algebra with the coactions $\beta : B \to C(G) \otimes B, \gamma : B \to C^*(G) \otimes B$ given by

$$\beta(T) = W^*(1 \otimes T)W, \qquad \gamma(T) = \hat{W}^*(1 \otimes T)W^*,$$

respectively.

The embeddings $M_n(\mathbb{C})^{\boxtimes k} \to M_n(\mathbb{C}) \boxtimes M_n(\mathbb{C})^{\boxtimes k} \cong M_n(\mathbb{C})^{\boxtimes k+1}$ given by $T \mapsto 1 \boxtimes T$ are G-equivariant.

The inductive limit action on $A = \varinjlim M_n(\mathbb{C})^{\boxtimes k} \cong M_{n^\infty}(\mathbb{C})$ has the Rokhlin property.

Rigidity

We say that $*$-homomorphisms $\phi, \psi : A \to B$ are *approximately unitarily equivalent* if there exists a sequence of unitaries $u_n \in \tilde{B}$ such that

$$\phi(a) = \lim_{n\to\infty} u_n \psi(a) u_n^*$$

for all $a \in A$. Write $\phi \approx_{\mathrm{u}} \psi$ in this case.

Theorem

Let G be a coexact compact quantum group. Let $\alpha, \beta : A \to C(G) \otimes A$ be two G-actions on a separable C^*-algebra A. Assume that both have the spatial Rokhlin property. Then $\alpha \approx_{\mathrm{u}} \beta$ as $*$-homomorphisms if and only if there exists an equivariant isomorphism $\theta : (A, \alpha) \to (A, \beta)$ which is approximately inner as an ordinary $*$-isomorphism.

Actions on $\mathcal{O}_2$

A C^*-algebra $\mathcal{D}$ is called *strongly selfabsorbing* if there exists an isomorphism $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the embedding $d \mapsto 1 \otimes d$.

Proposition

Let G be a compact quantum group and $\mathcal{D}$ a strongly self-absorbing C^*-algebra. Then there exists at most one conjugacy class of G-actions on $\mathcal{D}$ with the spatial Rokhlin property.

Lemma

Let G be a coamenable compact quantum group and let $\iota : C(G) \to \mathcal{O}_2$ be a unital embedding. Then $(\mathrm{id} \otimes \iota) \circ \Delta : C(G) \to C(G) \otimes \mathcal{O}_2$ is nuclear and full.

Actions on $\mathcal{O}_2$

Theorem

Let G be a coamenable compact quantum group. Then up to conjugacy, there exists a unique G-action on the Cuntz algebra $\mathcal{O}_2$ with the spatial Rokhlin property.

Let $\iota : C(G) \to \mathcal{O}_2$ be an embedding. Consider the unital $*$-homomorphism $\Phi_n : C(G) \otimes \mathcal{O}_2^{\otimes n} \to C(G) \otimes \mathcal{O}_2^{\otimes n+1}$ given by

$$\Phi_n(x \otimes y) = \big((\mathrm{id} \otimes \iota) \circ \Delta\big)(x) \otimes y$$

for $x \in C(G), y \in \mathcal{O}_2^{\otimes n}$. Each Φ_n is equivariant. Define

$$(A, \alpha) = \varinjlim \{(C(G) \otimes \mathcal{O}_2^{\otimes n}, \Delta \otimes \mathrm{id}_{\mathcal{O}_2^{\otimes n}}), \Phi_n\},$$

where α is the inductive limit coaction. Each term in our inductive system has the spatial Rokhlin property, hence A has the Rokhlin property too. Moreover Φ_n is nuclear and sends any nonzero element to a full element. It follows that the inductive limit A is separable, unital, simple, nuclear and $\mathcal{O}_2$-absorbing. This implies $A \cong \mathcal{O}_2$.