Instytut Matematyczny
Polskiej Akademii Nauk

RANK-TWO MILNOR IDEMPOTENTS FOR THE MULTIPULLBACK QUANTUM COMPLEX PROJECTIVE PLANE

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Joint work with
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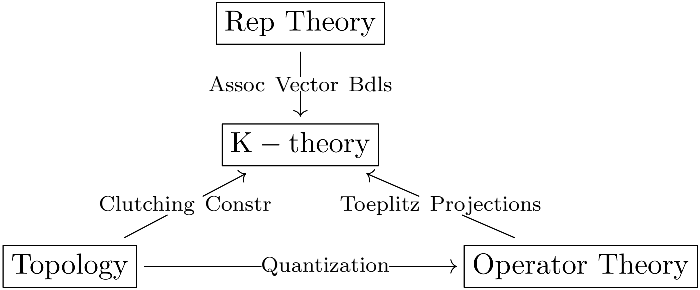
Road map


Rep Theory
Assoc Vector Bdls
K – theory
Clutching Constr
Toeplitz Projections
Topology
Quantization
Operator Theory

Free actions of compact quantum groups

Let A be a unital C^*-algebra and $\delta : A \to A \otimes_{\min} H$ an *injective* unital $*$-homomorphism. We call δ a coaction (or an action of the compact quantum group (H, Δ) on A) if

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H$ (counitality).

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Given a compact quantum group (H, Δ), we denote by $\mathcal{O}(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations.

The Peter-Weyl subalgebra

of A is $\mathcal{P}_H(A) := \{\, a \in A \,|\, \delta(a) \in A \otimes_{\mathrm{alg}} \mathcal{O}(H) \,\}$.

Pulling back noncommutative vector bundles

Theorem (P. M. H., T. Maszczyk)

Let (H, Δ) *be a compact quantum group,* A *and* A' (H, Δ)*-C*-algebras,* B *and* B' *the corresponding fixed-point subalgebras, and* $f : A \to A'$ *an equivariant *-homomorphism. Then, if the action of* (H, Δ) *on* A *is free and* V *is a representation of* (H, Δ)*, the following left* B'*-modules are isomorphic*

$$B'_f \underset{B}{\otimes} \big(\mathcal{P}_H(A)\Box V\big) \cong \mathcal{P}_H(A')\Box V \,.$$

Here B'_f *stands for the* B'*-*B*-bimodule with the right action given by* f*, i.e.* $b \cdot c = bf(c)$.

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Pulling-back Corollary

The induced map $(f|_B)_* : K_0(B) \to K_0(B')$ satisfies

$$(f|_B)_*\big([\mathcal{P}_H(A)\Box V]\big) = [\mathcal{P}_H(A')\Box V]\,.$$

Odd-to-even connecting homomorphism

For any one-surjective pullback diagram of rings

$$\begin{array}{ccccc} R_1 & \longleftarrow & R & \longrightarrow & R_2\ , \\ & \overset{\pi^1}{\searrow} & & \overset{\pi^2}{\swarrow} & \\ & & R_{12} & & \end{array}$$

there exists the following long exact sequence in algebraic K-theory:

$$\cdots \longrightarrow K_1^{\mathrm{alg}}(R_{12}) \xrightarrow{\partial_{10}^{\mathrm{alg}}} K_0(R) \longrightarrow K_0(R_1 \oplus R_2) \longrightarrow K_0(R_{12}),$$

with $\partial_{10}^{\mathrm{alg}}$ determined by $GL_\infty(R_{12}) \ni U \longmapsto M \in Proj(R)$,

$$\begin{array}{ccccc} R_1^n & \longleftarrow & M & \longrightarrow & R_2^n \\ & \overset{(\pi^1,\cdots,\pi^1)}{\searrow} & & \overset{(\pi^2,\cdots,\pi^2)}{\swarrow} & \\ & R_{12}^n & \xrightarrow{U} & R_{12}^n\ . & \end{array}$$

The Milnor idempotent

There exist liftings $c, d \in M_n(R_2)$ such that $\pi^2(c) = U$ and $\pi^2(d) = U^{-1}$. These liftings yield an explicit invertible lifting $\widetilde{U} \in M_{2n}(R_2)$ of $\left(\begin{smallmatrix} U & 0 \\ 0 & U^{-1} \end{smallmatrix}\right)$ as follows:

$$\widetilde{U} := \begin{pmatrix} c(2-dc) & cd-1 \\ 1-dc & d \end{pmatrix}, \text{ with } \widetilde{U}^{-1} = \begin{pmatrix} d & 1-dc \\ cd-1 & c(2-dc) \end{pmatrix}.$$

Now we can write an idempotent matrix (Milnor idempotent)

$$\boxed{p_U := \left(\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}, \ \widetilde{U} \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \widetilde{U}^{-1} \right) \in M_{2n}(R)}$$

representing the pullback module: $M \cong R^{2n} p_U$. Here I_n is the identity matrix of the same size as the matrix U.

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representing the pullback module: $M \cong R^{2n} p_U$. Here I_n is the identity matrix of the same size as the matrix U. The assignment

$$\boxed{\partial_{10}^{\mathrm{alg}} : K_1^{\mathrm{alg}}(R_{12}) \ni [U]_{\mathrm{alg}} \longmapsto [p_U] - [I_n] \in K_0(R)}$$

defines the odd-to-even connecting homomorphism.

The Mayer-Vietoris 6-term exact sequence

If R_{12} is a unital C*-algebra, there is a functorial surjection $K_1^{\mathsf{alg}}(R_{12}) \ni [U]_{\mathsf{alg}} \mapsto [U] \in K_1(R_{12})$. One can prove that any set-theoretical splitting $s\colon K_1(R_{12}) \to K_1^{\mathsf{alg}}(R_{12})$ defines a connecting homomorphism for the Mayer-Vietoris 6-term exact sequence of a one-surjective pullback of unital C*-algebras via the formula $\partial_{10} := \partial_{10}^{\mathrm{alg}} \circ s$.

$$
\begin{array}{ccccc}
K_0(R) & \longrightarrow & K_0(R_1) \oplus K_0(R_2) & \xrightarrow{\pi^1_* - \pi^2_*} & K_0(R_{12}) \\
\partial_{10} \uparrow & & & & \downarrow \partial_{01} \\
K_1(R_{12}) & \xleftarrow{\pi^1_* - \pi^2_*} & K_1(R_1) \oplus K_1(R_2) & \longleftarrow & K_1(R),
\end{array}
$$

Here the even-to-odd connecting homomorphism ∂_{01} is given by the formula

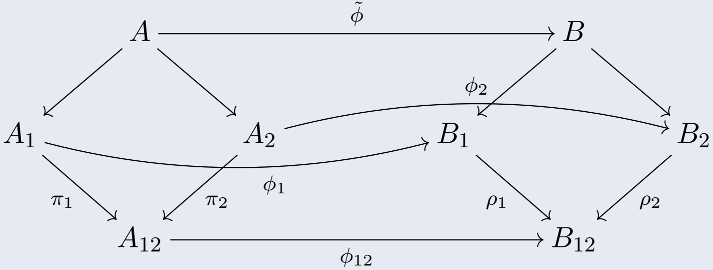
$$\boxed{\partial_{01}([p]) := [I_n, e^{2\pi i Q}]} \, ,$$

where Q is a self-adjoint lifting of the projection p to $M_n(R_2)$, i.e. $\pi^2(Q) = p$ and $Q^* = Q$.

K-Isomorphism Theorem

Theorem

*Let $\phi_1 : A_1 \to B_1$, $\phi_2 : A_1 \to B_2$ and $\phi_{12} : A_{12} \to B_{12}$ be *-homomorphisms between pullback diagrams of C*-algebras rendering the entire diagram*



*commutative and inducing isomorphisms on K-groups. Then, if π_2 and ρ_2 are surjective, the induced *-homomorphism $\tilde{\phi}$ also yields an isomorphim in K-theory.*

Modules associated to piecewise cleft coactions

Let $\mathcal{H}$ be a Hopf algebra, let

$$\begin{array}{ccccc} & & \mathcal{P} & & \\ & \swarrow & & \searrow & \\ \mathcal{P}_1 & \xrightarrow[\tilde{\pi}_1]{} & \mathcal{P}_{12} & \xleftarrow[\tilde{\pi}_2]{} & \mathcal{P}_2 \end{array}$$

be a one-surjective pullback diagram of $\mathcal{H}$-comodule algebras, and let $\gamma_i : \mathcal{H} \to \mathcal{P}_i$ be cleaving maps.

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Clutching Theorem

If V is a finite-dimensional left $\mathcal{H}$-comodule, then the associated left $\mathcal{P}_{12}^{\mathrm{co}\,\mathcal{H}}$-module $\mathcal{P} \Box V$ is the Milnor module for the automorphism of $\mathcal{P}_{12}^{\mathrm{co}\,\mathcal{H}} \otimes V$ given by

$$b \otimes v \longmapsto b(\tilde{\pi}_1 \circ \gamma_1)(v_{(-2)})(\tilde{\pi}_2 \circ \gamma_2)^{-1}(v_{(-1)}) \otimes v_{(0)}.$$

Quantum balls and spheres

For the Hong-Szymański quantum balls and Vaksman-Soibelman quantum spheres, we just proved:

Theorem (F. D'Andrea, P. M. H., M. Tobolski)

$\forall\, n \in \mathbb{N} \setminus \{0\}$ $\exists$ a $U(1)$-equivariant pullback of C-algebras:*

$$\begin{array}{ccc} & C(S_q^{2n+1}) & \\ \swarrow & & \searrow \\ C(S_q^{2n-1}) & & C(B_q^{2n}) \otimes C(S^1) \\ \searrow & & \swarrow \\ & C(S_q^{2n-1}) \otimes C(S^1). & \end{array}$$

Bundles over quantum complex projective spaces

Corollary

$\forall\, n \in \mathbb{N} \setminus \{0\}\ \exists$ *a pullback of C*-algebras:*

$$\begin{array}{ccccc} & & C(\mathbb{C}P_q^n) & & \\ & \swarrow & & \searrow & \\ C(\mathbb{C}P_q^{n-1}) & & & & C(B_q^{2n}) \\ & \searrow & & \swarrow & \\ & & C(S_q^{2n-1}). & & \end{array}$$

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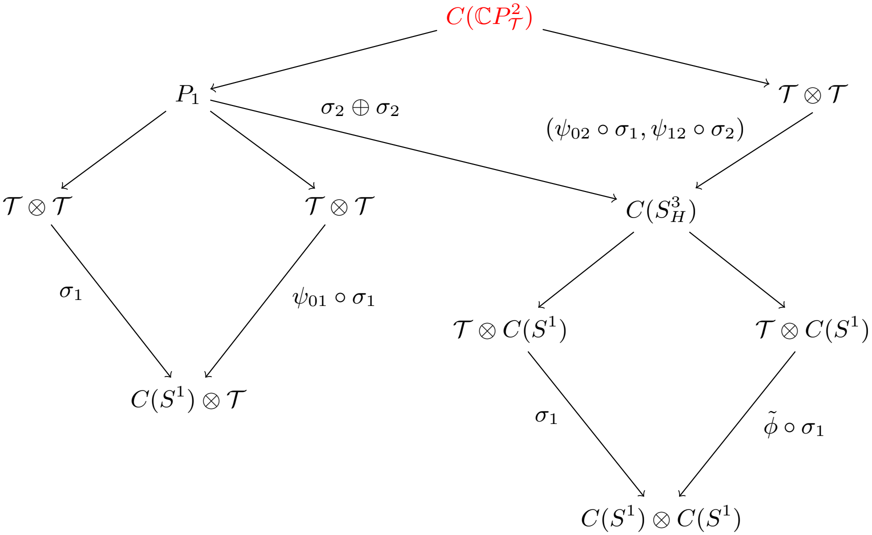
$$\begin{array}{ccccc} K_0(C(\mathbb{C}P_q^n)) & \longrightarrow & K_0(C(\mathbb{C}P_q^{n-1})) \oplus K_0(C(B_q^{2n})) & \longrightarrow & K_0(C(S_q^{2n-1})) \\ \partial_{10} \uparrow & & & & \downarrow \partial_{01} \\ K_1(C(S_q^{2n-1})) & \longleftarrow & K_1(C(\mathbb{C}P_q^{n-1})) \oplus K_1(C(B_q^{2n})) & \longleftarrow & K_1(C(\mathbb{C}P_q^n)) \end{array}$$

Multipullback quantum complex projective



Multipullback quantum spheres S_H^{2N+1}

$C(S_H^{2N+1})$ is the C*-subalgebra of $\prod_{i=0}^{N} \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i}$ defined by the compatibility conditions prescribed by the following diagrams ($0 \leq i < j \leq N$, $\otimes$-supressed):

$$\begin{array}{ccc} \mathcal{T}^i C(S^1) \mathcal{T}^{N-i} & & \mathcal{T}^j C(S^1) \mathcal{T}^{N-j} \\ & \searrow^{\sigma_j} \quad {}^{\sigma_i}\swarrow & \\ & \mathcal{T}^i C(S^1) \mathcal{T}^{j-i-1} C(S^1) \mathcal{T}^{N-j}. & \end{array}$$

Here $\sigma_k := \mathrm{id}^k \otimes \sigma \otimes \mathrm{id}^{N-k}$ with domains and codomains determined by the context.

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$$\mathcal{T}^i C(S^1)\mathcal{T}^{N-i} \xrightarrow{\sigma_j} \mathcal{T}^i C(S^1)\mathcal{T}^{j-i-1} C(S^1)\mathcal{T}^{N-j} \xleftarrow{\sigma_i} \mathcal{T}^j C(S^1)\mathcal{T}^{N-j}.$$

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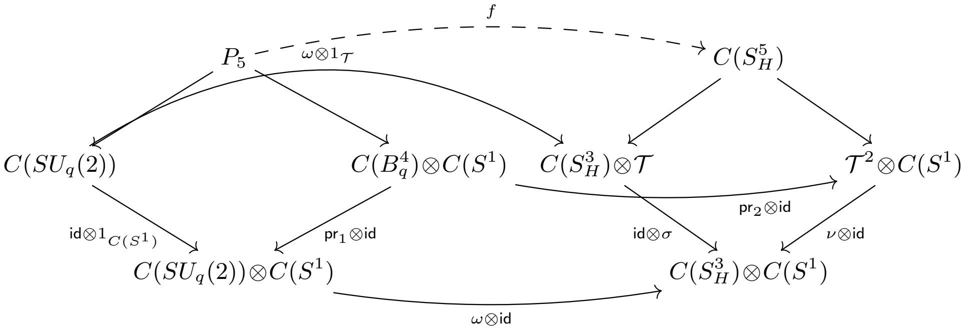
We equip all C*-algebras in the diagrams with the diagonal actions of $U(1)$. Since all morphisms in the diagrams are $U(1)$-equivariant, we obtain the diagonal $U(1)$-action on $C(S_H^{2N+1})$.

Reducing to the quantum-group case



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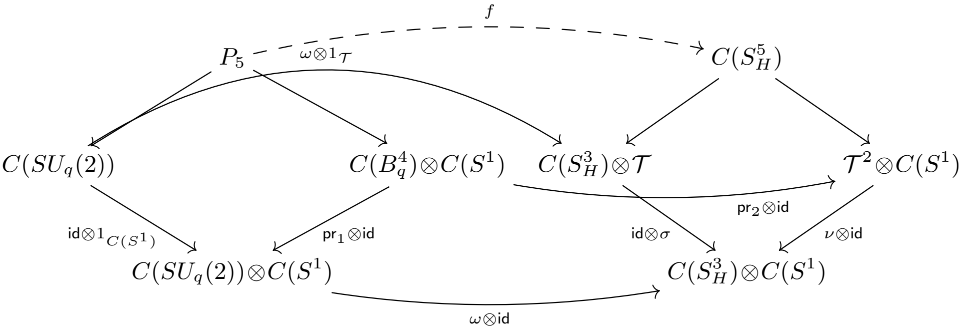

K-Isomorphism Lemma

The above *-homomorphisms are $U(1)$-equivariant, and the induced *-homomorphisms on fixed-point subalgebras yield isomorphisms on K-groups.

Main result

Definition

Let $k \in \mathbb{Z}$. We call the left $C(\mathbb{C}P^2_{\mathcal{T}})$-module

$$L_k := \{a \in C(S^5_H) \mid \forall\, \lambda \in U(1) : \alpha_\lambda(a) = \lambda^k a\}$$

the section module of the associated line bundle of winding number k.

Theorem

The group $K_0(C(\mathbb{C}P^2_{\mathcal{T}}))$ is freely generated by elements

$$[1], \quad [L_1] - [1], \quad [L_1 \oplus L_{-1}] - [2].$$

Furthermore, $L_1 \oplus L_{-1} \cong C(\mathbb{C}P^2_{\mathcal{T}}) \oplus C(\mathbb{C}P^2_{\mathcal{T}})e$. Here $e \in C(\mathbb{C}P^2_{\mathcal{T}})$ is an idempotent such that $C(\mathbb{C}P^2_{\mathcal{T}})e$ cannot be realized as a finitely generated projective module associated with the $U(1)$-C-algebra $C(S^5_H)$ of Heegaard quantum 5-sphere.*

Proof outline

1. Take the $SU_q(2)$-prolongation $P_5\square^{\mathcal{O}(U(1))}\mathcal{O}(SU_q(2))$. Then take the fundamental representation $\mathbb{C}^2$ of $SU_q(2)$, and compute the clutching matrix of the associated module
$$P_5\square^{\mathcal{O}(U(1))}\mathcal{O}(SU_q(2))\square^{\mathcal{O}((SU_q(2))}\mathbb{C}^2 = L'_1 \oplus L'_{-1}$$
from the Clutching Theorem.

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2. The clutching matrix turns out to be the fundamental representation matrix U_f, so its class generates $K_1(C(SU_q(2)))$. Now it follows from the six-term Mayer-Vietoris exact sequence that the Milnor class
$$\partial_{10}([U_f]) = [L'_1 \oplus L'_{-1}] - 2$$
is the third generator of $K_0(P_5^{U(1)})$.

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3. Finally, combining the above with the Isomorphism Lemma and the Pulling-back Corollary, we conclude that $[L_1 \oplus L_{-1}] - 2$ is the third generator of $K_0(C(\mathbb{C}P^2_\mathcal{T}))$.

From $K_0(C(\mathbb{T}^2))$ to $K_1(C(S^3_H))$

T. Loring proved that

$$\beta := \begin{pmatrix} 1 \otimes f & 1 \otimes g + u \otimes h \\ 1 \otimes g + u^* \otimes h & 1 \otimes 1 - 1 \otimes f \end{pmatrix}$$

generates the non-trivial part of $K_0(C(\mathbb{T}^2))$. Here u is the standard generating unitary of $C(S^1)$, and f, g and h are appropriately chosen functions on S^1.

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generates the non-trivial part of $K_0(C(\mathbb{T}^2))$. Here u is the standard generating unitary of $C(S^1)$, and f, g and h are appropriately chosen functions on S^1. Next, let s be the generating isometry of the Toeplitz algebra $\mathcal{T}$. Then

$$Q := \begin{pmatrix} 1 \otimes f & 1 \otimes g + s \otimes h \\ 1 \otimes g + s^* \otimes h & 1 \otimes 1 - 1 \otimes f \end{pmatrix} \in M_2(\mathcal{T} \otimes C(S^1))$$

is a self-adjoint lifting of β, and the even-to-odd connecting homomorphism yields a generator of $K_1(C(S^3_H)) \cong \mathbb{Z}$:

$$\partial_{01}([\beta]) = [(e^{2\pi i Q}, I_2)].$$

The Milnor idempotent simplified

One can explicitly compute $e^{2\pi i Q}$ to be:

$$e^{2\pi i Q} = \begin{pmatrix} 1\otimes 1 + (1-ss^*)\otimes(\exp(2\pi i\chi_{[0,\frac{1}{2}]}f)-1) & 0\otimes 0 \\ 0\otimes 0 & 1\otimes 1 \end{pmatrix},$$

where $\chi_{[0,\frac{1}{2}]}$ is a characterisitc function.

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where $\chi_{[0,\frac{1}{2}]}$ is a characterisitc function. Denote the upper left entry of the above matrix by v. Since

$$[(I_2, e^{2\pi iQ})] = [(1,v)] \in K_1(C(S^3_H)),$$

we can take $U = (1,v)$ to compute the Milnor idempotent p_U.

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$$[(I_2, e^{2\pi i Q})] = [(1, v)] \in K_1(C(S_H^3)),$$

we can take $U = (1, v)$ to compute the Milnor idempotent p_U. Finally, homotoping p_U we arrive at

$$[p_U] = \left[\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0(C(\mathbb{C}P^2_{\mathcal{T}})),$$

where

$$\boxed{e := (1, ss^* \otimes 1 + (1 - ss^*) \otimes ss^*)}.$$