The Plancherel Formula for complex quantum groups

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The classical Plancherel Theorem

For $f \in L^1(\mathbb{R})$ the Fourier transform of f is defined by

$$\mathcal{F}(f)(p) = \int_{\mathbb{R}} e^{-ixp} f(x) dx,$$

where dx denotes (suitably normalised) Lebesgue measure.

Theorem (Plancherel)

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then

$$\|\mathcal{F}(f)\|_2^2 = \int_{\mathbb{R}} |\mathcal{F}(f)(p)|^2 dp = \int_{\mathbb{R}} |f(x)|^2 dx = \|f\|_2^2.$$

Hence $\mathcal{F}$ induces a unitary isomorphism $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$.

The classical Plancherel Theorem

Let us reinterpret the Plancherel Theorem from a slightly more general perspective.

Since $\mathbb{R}$ is a locally compact abelian group, it has a Pontrjagin dual group $\hat{\mathbb{R}}$, consisting of all unitary characters of $\mathbb{R}$.

The unitary characters of $\mathbb{R}$ are of the form

$$\chi_p(x) = e^{-ixp}$$

for $p \in \mathbb{R}$.

In this way one obtains $\hat{\mathbb{R}} \cong \mathbb{R}$.

The classical Plancherel Theorem

The group C^*-algebra $C^*(\mathbb{R})$ is a completion of $C_c^\infty(\mathbb{R})$, equipped with the convolution product

$$(f * g)(t) = \int_{\mathbb{R}} f(-s)g(s+t)ds$$

and $*$-structure

$$f^*(t) = \overline{f(-s)}.$$

In particular, for the one-dimensional representations corresponding to the characters χ_p we obtain $*$-homomorphisms $\chi_p : C^*(\mathbb{R}) \to \mathbb{C}$ given by

$$\chi_p(f) = \int_{\mathbb{R}} f(x)\chi_p(x)dx = \int_{\mathbb{R}} f(x)e^{-ipx}dx = \mathcal{F}(f)(p)$$

for $f \in C_c^\infty(\mathbb{R})$.

The classical Plancherel Theorem

For $f \in C_c^\infty(\mathbb{R})$ we have

$$
\begin{aligned}
(f^* * f)(0) &= \int_{\mathbb{R}} \overline{f(s)} f(s) ds \\
&= \|f\|_2^2 = \|\mathcal{F}(f)\|_2^2 = \int_{\hat{\mathbb{R}}} \overline{\mathcal{F}(f)(p)} \mathcal{F}(f)(p) dp \\
&= \int_{\hat{\mathbb{R}}} \chi_p(f)^* \chi_p(f) dp = \int_{\hat{\mathbb{R}}} \chi_p(f^* * f) dp,
\end{aligned}
$$

or equivalently,

Theorem (Plancherel formula)

For any $h \in C_c^\infty(\mathbb{R})$ we have

$$
h(0) = \int_{\hat{\mathbb{R}}} \chi_p(h) dp.
$$

Plancherel versus Peter-Weyl

Now let G be a compact group.

Write $\mathrm{Irr}(G)$ for the set of equivalence classes of irreducible representations of G, and $\pi_\lambda : G \to U(\mathcal{H}_\lambda)$ for $\lambda \in \mathrm{Irr}(G)$.

Theorem (Peter-Weyl)

For $f \in L^1(G) \cap L^2(G)$ we have

$$\|f\|_2^2 = \sum_{\lambda \in \mathrm{Irr}(G)} \mathrm{tr}(\pi_\lambda(f)^* \pi_\lambda(f)) \dim(\mathcal{H}_\lambda)^{-1}$$

Plancherel versus Peter-Weyl

Hence the formula

$$\mathcal{F}(f) = \bigoplus_{\lambda \in \mathrm{Irr}(G)} \pi_\lambda(f)$$

for $f \in L^1(G) \cap L^2(G)$ extends to an isometric isomorphism

$$\mathcal{F} : L^2(G) \to \bigoplus_{\lambda \in \mathrm{Irr}(G)} HS(\mathcal{H}_\lambda),$$

if on $\mathrm{Irr}(G)$ we consider the (Plancherel) measure

$$dm = \sum_{\lambda \in \mathrm{Irr}(G)} \dim(\mathcal{H}_\lambda)^{-1} \delta_\lambda.$$

Abstract Plancherel Theorem

Assume that G is a type I locally compact possibly non-unimodular quantum group.

Theorem (Segal-Mautner, Duflo-Moore, Desmedt)

Then there exists a standard measure m on $\mathrm{Irr}(G)$, a measurable field of Hilbert spaces $(\mathcal{H}_\lambda)_{\lambda \in \mathrm{Irr}(G)}$, a measurable field $(D_\lambda)_{\lambda \in \mathrm{Irr}(G)}$ of self-adjoint strictly positive operators for $(\mathcal{H}_\lambda)_{\lambda \in \mathrm{Irr}(G)}$, and an isometric G-equivariant isomorphism

$$\mathcal{F} : L^2(G) \to \int_{\mathrm{Irr}(G)}^{\oplus} HS(\mathcal{H}_\lambda) dm(\lambda),$$

given by

$$\mathcal{F}(f) = \int_{\mathrm{Irr}(G)}^{\oplus} \pi_\lambda(f) D_\lambda^{-1} dm(\lambda)$$

on a dense subspace of $L^1(G) \cap L^2(G)$.

Remark on Duflo-Moore operators

The appearance of Duflo-Moore operators is not really due to non-unimodularity, but rather related to the question of whether the (left) Haar weight of the group algebra is a trace or not. In the group case, this is equivalent to (non-) unimodularity.

For instance, for a compact quantum group, there are Duflo-Moore operators in the Plancherel formula. These are trivial iff the quantum group is of Kac type - note that compact quantum groups are always unimodular.

If G is a compact quantum group the Plancherel formula becomes

$$\epsilon(f) = \sum_{\lambda \in \mathrm{Irr}(G)} \dim_q(\mathcal{H}_\lambda)\, \mathrm{tr}(\pi_\lambda(f) D_\lambda^{-2})$$

for $f \in \mathcal{O}(G)$.

Complex semisimple quantum groups

A little bit of history:

- Podleś-Woronowicz (1990) construct complex semisimple quantum groups on the C^*-algebra level.
- Pusz (1993), Pusz-Woronowicz (1994, 2000) completely classify the irreducible unitary representations of $SL_q(2, \mathbb{C})$.
- Buffenoir-Roche (1999) determine the Plancherel formula for $SL_q(2, \mathbb{C})$.
- Arano (2014, 2016) completely classifies the irreducible unitary representations of $SL_q(n, \mathbb{C})$, and most of the full dual in general.

Complex semisimple quantum groups

Here is a quick outline of the construction of the quantization G_q of a (simply connected) complex semisimple group G:

- Start from the Iwasawa decomposition $G = KAN$.
- For the compact part K there exists a deformation K_q obtained using quantized enveloping algebras.
- According to Drinfeld duality, a quantization of the Poisson dual AN of K is given by the Pontrjagin dual $\hat{K}_q$ of K_q.
- The complex quantum group G_q is the quantum double

$$G_q = K_q \bowtie \hat{K}_q.$$

We shall now explain the ingredients in these constructions in more detail.

Notation

- Fix $q = e^h \in (0,1)$.
- Let $\mathfrak{g}$ be a semisimple complex Lie algebra of rank N with Cartan matrix (a_{ij}).
- $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra.
- $\Delta = \Delta^+ \cup \Delta^-$ the root system with simple roots $\alpha_1, \dots, \alpha_N \subset \mathfrak{h}^*$.
- $(\, , \,)$ the bilinear form on $\mathfrak{h}^*$ obtained by rescaling the Killing form such that all short roots α satisfy $(\alpha, \alpha) = 2$.
- Set $d_i = (\alpha_i, \alpha_i)/2$ and $q_i = q^{d_i}$.
- $\varpi_1, \dots, \varpi_N \in \mathfrak{h}^*$ are the fundamental weights.
- $\mathbf{P} = \bigoplus_{j=1}^N \mathbb{Z}\varpi_j$ and $\mathbf{Q} = \bigoplus_{j=1}^N \mathbb{Z}\alpha_j$ are the weight and root lattices, respectively.
- $\mathbf{P}^+ = \bigoplus_{j=1}^N \mathbb{N}_0\varpi_j$ are the dominant integral weights.
- W is the Weyl group of $\mathfrak{g}$.

The Drinfeld-Jimbo algebra associated to $\mathfrak{g}$

The *quantized universal enveloping algebra* $U_q(\mathfrak{g})$ is the algebra with generators E_j, F_j for $1 \leq j \leq N$ and K_λ for $\lambda \in \mathbf{P}$ satisfying

$$K_0 = 1, \qquad K_\lambda K_\mu = K_{\lambda+\mu},$$

$$K_\lambda E_j K_\lambda^{-1} = q^{(\lambda,\alpha_j)} E_j, \qquad K_\lambda F_j K_\lambda^{-1} = q^{-(\lambda,\alpha_j)} F_j,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad \text{where } K_i = K_{\alpha_i},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0 \qquad i \neq j,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0 \qquad i \neq j.$$

The Drinfeld-Jimbo algebra associated to $\mathfrak{g}$

The algebra $U_q(\mathfrak{g})$ is a Hopf algebra.

For instance, the coproduct $\hat{\Delta} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is given by

$$\begin{aligned} \hat{\Delta}(K_\lambda) &= K_\lambda \otimes K_\lambda, \\ \hat{\Delta}(E_i) &= E_i \otimes K_i + 1 \otimes E_i \\ \hat{\Delta}(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i. \end{aligned}$$

Moreover $U_q(\mathfrak{g})$ is a $*$-algebra with the $*$-structure

$$E_i^* = K_i F_i, \qquad F_i^* = E_i K_i^{-1}, \qquad K_\lambda^* = K_\lambda.$$

As a Hopf $*$-algebra, $U_q(\mathfrak{g})$ should be viewed as quantization of the (complex) universal enveloping algebra of the (real) Lie algebra $\mathfrak{k}$.

Representation theory and representative functions

The finite dimensional representation theory of $U_q(\mathfrak{g})$ is similar to the one for $U(\mathfrak{g})$. In particular, for every $\mu \in \mathbf{P}^+$ there exists a unique irreducible representation $V(\mu)$ with a highest weight vector v_μ, satisfying

$$K_\lambda v_\mu = q^{(\lambda,\mu)} v_\mu$$

Using the representations $V(\mu)$ one defines a compact quantum group K_q as follows.

Definition

The algebra $\mathcal{O}(K_q) \subset U_q(\mathfrak{g})^*$ of representative functions on K_q is the Hopf $*$-algebra of matrix coefficients of all $V(\mu)$ for $\mu \in \mathbf{P}^+$. We let $C(K_q)$ be its universal C^*-completion.

$\mathcal{O}(K_q)$ is a deformation of the algebra $\mathcal{O}(K)$ of representative functions on K, and $C(K_q)$ is a deformation of $C(K)$.

Example: the quantum group $SU_q(2)$

The algebra $\mathcal{O}(SU_q(2))$ can be identified with the $*$-algebra generated by elements α and γ satisfying the relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma,$$
$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

These relations are equivalent to saying that the fundamental matrix

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is unitary.

The maximal torus survives the deformation untouched: There exists a $*$-homomorphism $\pi : \mathcal{O}(SU_q(2)) \to \mathcal{O}(T) = \mathbb{C}[z, z^{-1}]$ given by $\pi(\alpha) = z, \pi(\gamma) = 0$.

The quantization of AN

Every (locally compact) quantum group admits a Pontrjagin dual (locally compact) quantum group.

In the case of K_q, the dual $\hat{K}_q$ is encoded by the $*$-algebra

$$C_c(\hat{K}_q) = \mathcal{D}(K_q) = \bigoplus_{\mu \in \mathbf{P}^+} \mathrm{End}(V(\mu)),$$

equipped with a suitable coproduct.

To the classical group A corresponds the *quotient* $\hat{T}$ of $\hat{K}_q$ obtained from the projection $\mathcal{O}(K_q) \to \mathcal{O}(T)$. Here $T \subset K_q$ is the classical maximal torus.

Complex semisimple quantum groups

Consider the vector space

$$\mathcal{D}(G_q) = \mathcal{D}(K_q) \bowtie \mathcal{O}(K_q),$$

equipped with the multiplication

$$(x \bowtie f)(y \bowtie g) = x(f_{(1)}, y_{(1)})y_{(2)} \bowtie f_{(2)}(f_{(3)}, \hat{S}(y_{(3)}))g$$

and the $*$-structure

$$(x \bowtie f)^* = (1 \bowtie f^*)(x^* \bowtie 1).$$

Definition

The group C^*-algebra $C^*(G_q)$ of the complex quantum group G_q is the universal C^*-completion of $\mathcal{D}(G_q)$.

The representation theory of G_q

This leads to some natural tasks/questions.

- Describe all irreducible representations of G_q up to isomorphism.
- Describe the (reduced) unitary dual of G_q.
- Describe the Plancherel formula.
- Describe the Fell topology of the (reduced) dual.

The representation theory of G_q

By construction, a nondegenerate representation of $C^*(G_q)$ on a Hilbert space $\mathcal{H}$ corresponds to a nondegenerate $*$-homomorphism $\mathcal{D}(G_q) \to \mathcal{L}(\mathcal{H})$.

This is the same thing as a unitary *Yetter-Drinfeld module*, that is, a pair of a unital $*$-homomorphism $\mathcal{O}(K_q) \to \mathcal{L}(\mathcal{H})$ and a unitary corepresentation $V \in M(C(K_q) \otimes \mathcal{H})$ satisfying the Yetter-Drinfeld compatibility condition, given by

$$f_{(1)}\xi_{(-1)}S(f_{(3)}) \otimes f_{(2)} \cdot \xi_{(0)} = (f \cdot \xi)_{(-1)} \otimes (f \cdot \xi)_{(0)}$$

for $f \in \mathcal{O}(K_q)$ and ξ in (a certain dense subspace of) $\mathcal{H}$.

Principal series representations

Let $\mathcal{O}(\mathcal{E}_\mu) \subset \mathcal{O}(K_q)$ be the spectral subspace of $\mathcal{O}(K_q)$ associated to $\mu \in \mathbf{P}$ with respect to the right action of T.

For $\lambda \in \mathfrak{h}^*$ we define the twisted left adjoint representation of $\mathcal{O}(K_q)$ on $\mathcal{O}(\mathcal{E}_\mu)$ by

$$f \cdot \xi = f_{(1)} \, \xi \, S(f_{(3)})(K_{\lambda+2\rho}, f_{(2)}).$$

Together with the comultiplication of $\mathcal{O}(K_q)$ this turns $\mathcal{O}(\mathcal{E}_\mu)$ into a Yetter-Drinfeld module, which we will denote by $\mathcal{O}(\mathcal{E}_{\mu,\lambda})$.

This is called the *principal series Yetter-Drinfeld module* with parameter $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}^*$.

If $\lambda \in i\mathfrak{a}^* \subset \mathfrak{h}^*$ then this Yetter-Drinfeld module is unitary. It corresponds to a representation of $C^*(G_q)$ on the Hilbert space completion of $\mathcal{O}(\mathcal{E}_\mu)$.

For $\lambda \in \mathfrak{h}^*$, the operators K_λ are defined by $K_\lambda v = q^{(\lambda,\nu)} v$.

Recall that $q = e^h$, and let $\hbar = \frac{h}{2\pi}$.

In particular, $K_\lambda = K_{\lambda'}$ if $\lambda - \lambda' \in i\hbar^{-1}\mathbf{Q}^\vee$. Here $\mathbf{Q}^\vee$ is the coroot lattice.

Hence, by their very construction, the principal series modules $\mathcal{O}(\mathcal{E}_{\mu,\lambda})$ and $\mathcal{O}(\mathcal{E}_{\mu,\lambda'})$ are *the same* if $\lambda - \lambda' \in i\hbar^{-1}\mathbf{Q}^\vee$.

Write

$$\mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}, \qquad \mathfrak{a}_q^* = \mathfrak{a}^*/\hbar^{-1}\mathbf{Q}.$$

This notation allows us to remove the "obvious" redundancies in the parametrisation of the principal series explained above.

The structure of principal series representations

For $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta$ write $\lambda_\alpha = 2(\alpha, \lambda)/(\alpha, \alpha)$.

Theorem

Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ such that $\lambda_\alpha \neq \pm(|\mu_\alpha| + 2j)$ modulo $i\hbar^{-1}\mathbb{Z}$ for all $j \in \mathbb{N}$ and all $\alpha \in \Delta^+$. Then the principal series module with parameter (μ, λ) is an irreducible Yetter-Drinfeld module.

Theorem

Let $(\mu, \lambda) \in \mathbf{P} \times i\mathfrak{t}_q^*$. Then the principal series modules with parameters (μ, λ) and (μ', λ') are equivalent iff $(\mu', \lambda') = (w\mu, w\lambda)$ for some $w \in W$.

These results are (essentially) due to Joseph-Letzter and depend on deep facts about the structure of $U_q(\mathfrak{g})$.

Theorem

Let $q \in (0,1)$ and let G_q be a complex semisimple quantum group. Moreover let $\mathcal{H} = (\mathcal{H}_{\mu,i\nu})_{\mu,\nu}$ be the Hilbert space bundle of unitary principal series representations over $\mathbf{P} \times \mathfrak{a}_q^*$. Then there is a unitary isomorphism

$$Q : L^2(G_q) \cong \bigoplus_{\mu \in \mathbf{P}} \int_{\nu \in \mathfrak{a}_q^*}^{\oplus} HS(\mathcal{H}_{\mu,i\nu}) dm_\mu(\nu)$$

for the measures dm_μ on $\mathfrak{a}_q^*$ given by

$$dm_\mu(\nu) = \prod_{\alpha \in \Delta^+} (q_\alpha^{1/2} - q_\alpha^{-1/2})^2 [(\mu + i\nu)_\alpha]_{q_\alpha^{1/2}} [(\mu - i\nu)_\alpha]_{q_\alpha^{1/2}} \, d\nu,$$

where $d\nu$ denotes normalised Lebesgue measure on $\mathfrak{a}_q^*$.

Some remarks

The proof proceeds by verifying the Plancherel formula

$$\epsilon_{G_q}(f) = \sum_{\mu \in \mathbf{P}} \int_{\mathfrak{a}_q^*} \mathrm{tr}(\pi_{\mu,i\nu}(f) D_{\mu,i\nu}^{-2}) dm_\mu(\nu)$$

for elements of the form $f = u_{ij}^{\beta} \otimes \omega_{kl}^{\gamma} \in \mathcal{O}(K_q) \otimes \mathcal{D}(K_q)$.

For this one starts by directly calculating the characters of principal series representations.

In this computation, the universal R-matrix of $U_q(\mathfrak{g})$ enters crucially.

The lowest order contribution in h of the quantum Plancherel measure agrees with the classical Plancherel measure

$$\prod_{\alpha \in \acute{}+} |(\mu_\alpha + i\nu_\alpha)|^2 d\nu = (\mu + i\nu)_\alpha (\mu - i\nu)_\alpha d\nu$$

on $\mathbf{P} \times \mathfrak{a}^*$.

The reduced dual of G_q

The reduced group C^*-algebra of G_q is the norm closure of $\mathcal{D}(G_q)$ inside $\mathcal{L}(L^2(G_q))$ under the regular representation.

Theorem

Let $q \in (0,1)$ and let G_q be a complex semisimple quantum group. Moreover let $\mathcal{H} = (\mathcal{H}_{\mu,\lambda})_{\mu,\lambda}$ be the Hilbert space bundle of principal series representations of G_q over $\mathbf{P} \times \mathfrak{a}_q^*$. Then the canonical $*$-homomorphism

$$\pi : C^*_\mathrm{r}(G_q) \to C_0(\mathbf{P} \times \mathfrak{a}_q^*, \mathbb{K}(\mathcal{H}))^W$$

is an isomorphism.

Setting formally $h = 0$ here (corresponding to $q = 1$), and $\mathfrak{a}_1^* = \mathfrak{a}^*$ one obtains the corresponding statement for the classical reduced group C^*-algebra $C^*_\mathrm{r}(G)$.

Baum-Connes

The deformation picture of the Baum-Connes assembly map for the classical complex group G provides an isomorphism

$$K_*(C^*(K \ltimes_{\mathrm{ad}} \mathfrak{k}^*)) = K_*(K \ltimes_{\mathrm{ad}} C_0(\mathfrak{k})) \to K_*(C^*_{\mathrm{r}}(G)).$$

Let us restrict attention to the case $G = SL(2, \mathbb{C})$.

Theorem

Fix $q \in (0,1)$. Then there is a commutative diagram

$$\begin{array}{ccc} K_*(K \ltimes_{\mathrm{ad}} C_0(\mathfrak{k})) & \xrightarrow{\mu} & K_*(C^*_{\mathrm{r}}(G)) \\ \downarrow & & \downarrow \\ K_*(K \ltimes_{\mathrm{ad}} C(K)) & \xrightarrow{\mu_q} & K_*(C^*_{\mathrm{r}}(G_q)) \end{array}$$

Both vertical maps are split injective, and the horizontal maps are isomorphisms.