THE K-THEORY OF TWISTED MULTIPULLBACK QUANTUM ODD SPHERES AND COMPLEX PROJECTIVE SPACES

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Finite free distributive lattices

By Koichi YAMAMOTO

(Received March 9, 1954)

1.—Introduction.—The problem to determine the order $f(n)$ of the free distributive lattice $FD(n)$ generated by n symbols $\gamma_1, \cdots, \gamma_n$ was first proposed by Dedekind, but very little is known about this number [1, p. 146]. Only the first six values of $f(n)$ are computed, and enumerations of further $f(n)$ appear to lie beyond the scope of any reasonable methods known today. It might, however, be pointed out that Morgan Ward, who found $f(6)$ by the help of computing machines, stated [2] an asymptotic relation

$$\log_2 \log_2 f(n) \sim n$$

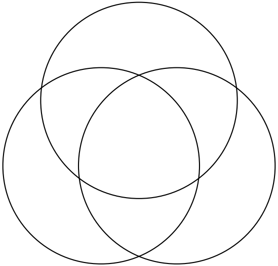
and that the present author proved in a previous note [3] that

$$f(n) \equiv 0 \pmod 2 \qquad \text{if} \qquad n \equiv 0 \pmod 2 .$$

An inspection of numerical results $f(n)$, $n \leqq 6$ suggests strongly the following asymptotic equivalence

$$(*) \qquad \log_2 f(n) \sim \sqrt{\frac{2}{\pi}}\, 2^n n^{-\frac{1}{2}} .$$

A classical model of a FDA



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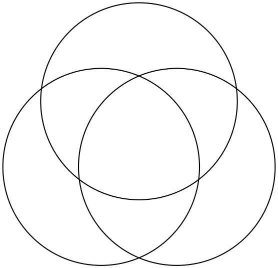

Consider the family $\{V_i\}_{i\in\{0,\ldots,N\}}$ of closed subsets of $\mathbb{P}^N(\mathbb{C})$ covering of $\mathbb{P}^N(\mathbb{C})$:

$$V_i := \{[x_0 : \ldots : x_N] \mid |x_i| = \max\{|x_0|, \ldots, |x_N|\}\}.$$

The distributive lattice generated by the subsets $V_i \subset \mathbb{P}^N(\mathbb{C})$ is free.

A noncommutative model of a FDA

Theorem (P.M.H., A. Kaygun, B. Zieliński)

Let $C(\mathbb{P}^N(\mathcal{T})) \subset \prod_{i=0}^{N} \mathcal{T}^{\otimes N}$ *be the C*-algebra of the Toeplitz quantum projective space, and let*

$$\pi_i \colon C(\mathbb{P}^N(\mathcal{T})) \longrightarrow \mathcal{T}^{\otimes N}, \quad i \in \{0, \ldots, N\},$$

be the family of restrictions of the canonical projections onto the components. Then the family of ideals $\{\ker \pi_i\}_{i\in\{0,\ldots,N\}}$ *generates a free distributive lattice.*

Odd-dimensional spheres from solid tori

$$\boxed{S^{2N+1} := \left\{(z_0, \dots, z_N) \in \mathbb{C}^{N+1} \mid |z_0|^2 + \cdots + |z_N|^2 = 1\right\}}$$

Let $V_i := \left\{(z_0, \dots, z_N) \in S^{2N+1} \mid |z_i| = \max\{|z_0|, \dots, |z_N|\}\right\}$.
Then

$$S^{2N+1} := \bigcup_{i=0}^{N} V_i \, .$$

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Then

$$S^{2N+1} := \bigcup_{i=0}^{N} V_i \, .$$

Homeomorphism implementing $V_i \cong D^{\times^i} \times S^1 \times D^{\times^{N-i}}$

$$\phi_i : V_i \ni (z_0, \dots, z_N) \longmapsto \left(\frac{z_0}{|z_i|}, \dots, \frac{z_N}{|z_i|}\right) \in D^{\times^i} \times S^1 \times D^{\times^{N-i}},$$

$$\phi_i^{-1} : D^{\times^i} \times S^1 \times D^{\times^{N-i}} \ni (d_0, \dots, d_{i-1}, c, d_{i+1}, \dots, d_N)$$
$$\longmapsto \frac{1}{\sqrt{1 + \sum_{j \neq i} |d_j|^2}} (d_0, \dots, d_{i-1}, c, d_{i+1}, \dots, d_N) \in V_i \, .$$

$C(S^{2N+1})$ as a multi-pullback C*-algebra

Definition

The multi-pullback algebra A^π of a finite family $\{\pi^i_j : A_i \longrightarrow A_{ij} = A_{ji}\}_{i,j\in J,\, i\neq j}$ of algebra morphisms is defined as

$$A^\pi := \left\{ (a_i)_{i\in J} \in \prod_{i\in J} A_i \;\middle|\; \pi^i_j(a_i) = \pi^j_i(a_j),\ \forall\, i,j \in J,\, i \neq j \right\}.$$

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$C(S^{2N+1})$ is isomorphic as a C*-algebra to the subalgebra of

$$\prod_{0\leq i\leq N} C(D)^{\otimes i} \otimes C(S^1) \otimes C(D)^{\otimes N-i}$$

defined by the compatibility conditions ($0 \leq i < j \leq N$, $\otimes$ suppressed):

$$C(D)^i C(S^1) C(D)^{N-i} \xrightarrow{\ \pi^i_j\ } C(D)^i C(S^1) C(D)^{j-i-1} C(S^1) C(D)^{N-j} \xleftarrow{\ \pi^j_i\ } C(D)^j C(S^1) C(D)^{N-j}.$$

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We have a short exact sequence of $U(1)$-equivariant C*-homomorphisms:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \stackrel{\sigma}{\longrightarrow} C(S^1) \longrightarrow 0.$$

Here u is the unitary generator of $C(S^1)$, $\mathcal{K}$ is the ideal of compact operators, and σ is the symbol map ($\sigma(z) := u$). The action α of $U(1)$ on $\mathcal{T}$ is given by $z \mapsto \lambda z$.

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We dualize this action to a coaction of $C(U(1))$ on $\mathcal{T}$. Explicitly, we have:

$$\rho : \mathcal{T} \longrightarrow \mathcal{T} \otimes C(U(1)) = C(U(1), \mathcal{T}),$$
$$\rho(t)(\lambda) := \alpha_\lambda(t), \quad \rho(z)(\lambda) = \lambda z, \quad \rho(z) = z \otimes u.$$

We use the Heyneman-Sweedler notation $\rho(t) =: t_{(0)} \otimes t_{(1)}$.

Quantum Dynamics, 2016–2019

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Tentative plan of conferences

New Geometry of Quantum Dynamics conferences pending approval:

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Multi-pullback quantum spheres S_H^{2N+1}

$C(S_H^{2N+1})$ is the C*-subalgebra of $\prod_{i=0}^{N} \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i}$ defined by the compatibility conditions prescribed by the following diagrams ($0 \leq i < j \leq N$, $\otimes$-supressed):

$$\mathcal{T}^i C(S^1)\mathcal{T}^{N-i} \xrightarrow{\sigma_j} \mathcal{T}^i C(S^1)\mathcal{T}^{j-i-1} C(S^1)\mathcal{T}^{N-j} \xleftarrow{\sigma_i} \mathcal{T}^j C(S^1)\mathcal{T}^{N-j}.$$

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We equip all C*-algebras in the diagrams with the diagonal actions of $U(1)$. Since all morphisms in the diagrams are $U(1)$-equivariant, we obtain the diagonal $U(1)$-action on $C(S_H^{2N+1})$.

Gauging coactions

Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a compact Hausdorff group G on a unital C*-algebra A. As with $U(1)$ acting on $\mathcal{T}$, we encode the G-action on A through the $C(G)$-coaction on A:

$$\rho : A \ni a \longmapsto a_{(0)} \otimes a_{(1)} \in A \otimes C(G) = C(G, A), \quad \rho(a)(g) := \alpha_g(a).$$

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G-equivariant C*-algebra isomorphisms:

$$F : (A \otimes C(G))^D \to (A \otimes C(G))^R, \quad a \otimes h \mapsto a_{(0)} \otimes a_{(1)} h,$$
$$F^{-1} : (A \otimes C(G))^R \to (A \otimes C(G))^D, \quad a \otimes h \mapsto a_{(0)} \otimes S(a_{(1)}) h.$$

Here $S(h)(g) := h(g^{-1})$.

$C(S_H^{2N+1})$ as a gauged multi-pullback

The following diagrams ($0 \leq i < j \leq N$, $\otimes$ suppressed) are $U(1)$-equivariant with respect to the $U(1)$-actions on the rightmost factors.

$$\begin{array}{ccccc} i & \mathcal{T}^N C(S^1) & & \mathcal{T}^N C(S^1) & j \\ & \downarrow{\scriptstyle \sigma_{j-1}} & & \downarrow{\scriptstyle \sigma_i} & \\ & \mathcal{T}^{j-1}C(S^1)\mathcal{T}^{N-j}C(S^1) & \xleftarrow{\tilde{\Psi}_{ij}} & \mathcal{T}^{i}C(S^1)\mathcal{T}^{N-i-1}C(S^1), & \end{array}$$

$$\tilde{\Psi}_{ij} : \bigotimes_{k=0}^{i-1} t_k \otimes v \otimes \bigotimes_{\substack{l=i+1 \\ l\neq j}}^{N} t_l \otimes w$$

$$\longmapsto \bigotimes_{\substack{k=0 \\ k\neq i}}^{j-1} t_{k(0)} \otimes S\left(\prod_{\substack{m=0 \\ m\neq i,j}}^{N} t_{m(1)}\right) S(v)w_{(1)} \otimes \bigotimes_{l=j+1}^{N} t_{l(0)} \otimes w_{(2)}.$$

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$C(S_H^{2N+1})$ is isomorphic as a $U(1)$-C*-algebra to the multi-pullback $U(1)$-C*-algebra of the above diagrams.

$C(\mathbb{P}^N(\mathcal{T}))$ is the C*-subalgebra of $\prod_{i=0}^{N} \mathcal{T}^{\otimes N}$ defined by the compatibility conditions prescribed by the diagrams $(0 \leq i < j \leq N)$:

$$
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& \mathcal{T}^{\otimes j-1} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-j} & \xleftarrow{\Psi_{ij}} & \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i-1}, &
\end{array}
$$

$$
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$$

Quantum complex projective spaces $\mathbb{P}^N(\mathcal{T})$

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It follows from the gauged presentation of $C(S_H^{2N+1})$ that $C(\mathbb{P}^N(\mathcal{T})) \cong C(S_H^{2N+1})^{U(1)}$.

Universal presentation of $C(S^{2N+1}_{H,\theta})$

Let us define the following elements of $C(S^{2N+1}_H)$:

$$a_i := \big((\sigma\otimes\mathrm{id}^{\otimes N})(1^{\otimes i}\otimes z\otimes 1^{\otimes N-i}), \ldots, (\mathrm{id}^{\otimes N}\otimes\sigma)(1^{\otimes i}\otimes z\otimes 1^{\otimes N-i})\big).$$

It straightforward to check that $\forall\, i,j \in \{0,\ldots,N\},\ i\neq j:$

$$a_ia_j = a_ja_i, \quad a_ia_j^* = a_j^*a_i, \quad a_i^*a_i = 1, \quad \prod_{i=0}^{N}(1-a_ia_i^*) = 0.$$

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Lemma (Key Lemma)

$C(S^{2N+1}_H)$ is isomorphic as a $U(1)$-C-algebra with the universal C*-algebra generated by a_i's satisfying the above relations. The $U(1)$-action on the latter is given by rephasing the generators.*

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$C(S^{2N+1}_H)$ *is isomorphic as a* $U(1)$*-C*-algebra with the universal C*-algebra generated by* a_i*'s satisfying the above relations. The* $U(1)$*-action on the latter is given by rephasing the generators.*

Corollary

$C(S^{2N+1}_H)\cong\mathcal{T}^{\otimes N+1}/\mathcal{K}^{\otimes N+1}$, $K_0(C(S^{2N+1}_{H,\theta}))=\mathbb{Z}[C(S^{2N+1}_{H,\theta})]=\mathbb{Z}$, $K_1(C(S^{2N+1}_{H,\theta}))=\mathbb{Z}$.

A key exact sequence

Lemma

With respect to the diagonal $U(1)$-action, for any positive integer k, there exists a $U(1)$-equivariant short exact sequence of C^*-algebras

$$0 \longrightarrow C(S_H^{2k-1}) \otimes \mathcal{K} \longrightarrow C(S_H^{2k+1}) \longrightarrow \mathcal{T}^{\otimes k} \otimes C(S^1) \longrightarrow 0.$$

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Proof.

$$0 \longrightarrow \mathcal{T}^{\otimes k} \otimes \mathcal{K} \longrightarrow \mathcal{T}^{\otimes k} \otimes \mathcal{T} \xrightarrow{\mathrm{id} \otimes \sigma} \mathcal{T}^{\otimes k} \otimes C(S^1) \longrightarrow 0 \,,$$

$$\big(\mathcal{T}^{\otimes k} \otimes \mathcal{K}\big)/\mathcal{K}^{\otimes k+1} \cong C(S_H^{2k-1}) \otimes \mathcal{K},$$

$$\big(\mathcal{T}^{\otimes k} \otimes \mathcal{T}\big)/\mathcal{K}^{\otimes k+1} \cong C(S_H^{2k+1}).$$

□

Invariant subalgebras

For all $k \in \{1, \ldots, N\}$, we have

$$0 \longrightarrow C(S_H^{2k-1}) \otimes \mathcal{K}^{\otimes N-k+1} \longrightarrow C(S_H^{2k+1}) \otimes \mathcal{K}^{\otimes N-k}$$
$$\longrightarrow \mathcal{T}^{\otimes k} \otimes C(S^1) \otimes \mathcal{K}^{\otimes N-k} \longrightarrow 0.$$

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Next, let

$$S_k := \left(C(S_H^{2k+1}) \otimes \mathcal{K}^{\otimes N-k}\right)^{U(1)}, \quad k \in \{0, \dots, N\}.$$

Using this notation we can write

$$0 \longrightarrow S_{k-1} \longrightarrow S_k \longrightarrow \mathcal{T}^{\otimes k} \otimes \mathcal{K}^{\otimes N-k} \longrightarrow 0,$$

where $k \in \{1, \dots, N\}$.

K-groups

Theorem

$\forall\, N \in \mathbb{N} \setminus \{0\} \colon K_0(C(\mathbb{P}^N_\theta(\mathcal{T}))) = \mathbb{Z}^{N+1}$ *and* $K_1(C(\mathbb{P}^N_\theta(\mathcal{T}))) = 0.$

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Proof. We prove by induction that $K_0(S_k) = \mathbb{Z}^{k+1}$ and $K_1(S_k) = 0$ for all $k \in \{0, \ldots, N\}$. The first step follows from $S_0 = \mathcal{K}$, the induction step follows from

$$\begin{array}{ccccc} K_0(S_{k-1}) & \longrightarrow & K_0(S_k) & \longrightarrow & K_0(\mathcal{T}^{\otimes k}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{T}^{\otimes k}) & \longleftarrow & K_1(S_k) & \longleftarrow & K_1(S_{k-1}), \end{array}$$

and the conclusion follows from $S_N = C(\mathbb{P}^N(\mathcal{T}))$. □

Noncommutative line bundles

Theorem

Let $L_k^{2N+1} := \{a \in C(S_H^{2N+1}) \mid \forall\, \lambda \in U(1) : \alpha_\lambda(a) = \lambda^k a\}$. *Then* $\forall\, N \in \mathbb{N} \setminus \{0\}$: $[L_m^{2N+1}] = [L_n^{2N+1}] \implies m = n$.

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Proof outline:

1. By Key Lemma, the assignments $a_k \mapsto b_k$ when $k < 2$ and $a_k \mapsto b_0$ when $k \geq 2$ define a $U(1)$-equivariant C*-homomorphism $f : C(S_H^{2N+1}) \to C(S_H^3)$. Here $a_0, \ldots, a_N$ are isometries generating $C(S_H^{2N+1})$ and b_0, b_1 are isometries generating $C(S_H^3)$.

Noncommutative line bundles

Theorem

Let $L_k^{2N+1} := \{a \in C(S_H^{2N+1}) \mid \forall\, \lambda \in U(1) : \alpha_\lambda(a) = \lambda^k a\}$*. Then* $\forall\, N \in \mathbb{N} \setminus \{0\}$: $[L_m^{2N+1}] = [L_n^{2N+1}] \implies m = n$*.*

Proof outline:

1. By Key Lemma, the assignments $a_k \mapsto b_k$ when $k < 2$ and $a_k \mapsto b_0$ when $k \geq 2$ define a $U(1)$-equivariant C*-homomorphism $f : C(S_H^{2N+1}) \to C(S_H^3)$. Here $a_0, \ldots, a_N$ are isometries generating $C(S_H^{2N+1})$ and b_0, b_1 are isometries generating $C(S_H^3)$.
2. Taking advantage of Chern-Galois theory [Brzeziński, P.M.H.], we conclude that the induced map

$$f_* : K_0(C(\mathbb{P}^N(\mathcal{T}))) \longrightarrow K_0(C(\mathbb{P}^1(\mathcal{T})))$$

satisfies $f_*([L_m^{2N+1}]) = [L_m^3]$ for any $m \in \mathbb{Z}$.

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3. Finally, as an index pairing computation proves that $[L_m^3] = [L_n^3] \Rightarrow m = n$ [P.M.H., R. Matthes, W. Szymański], the conclusion follows.