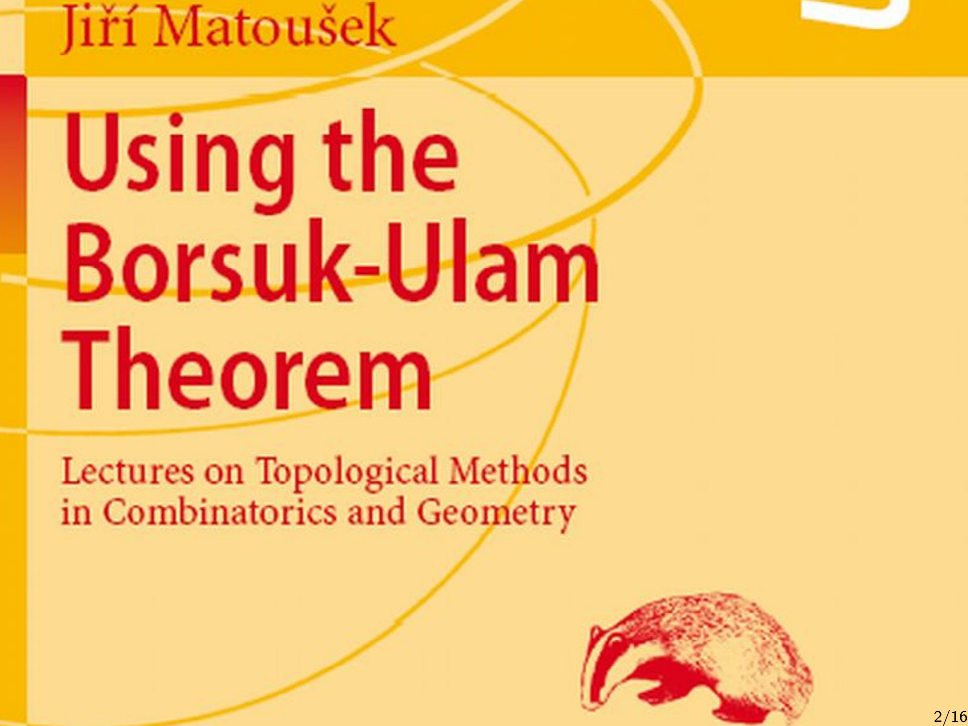
09 kesäkuu 2016

UNIVERSITY OF OULU
OULUN YLIOPISTO

THERE AND BACK AGAIN: FROM THE BORSUK-ULAM THEOREM TO QUANTUM SPACES

Piotr M. Hajac (IMPAN / University of New Brunswick)

Perustuen yhteistä työtä kanssa
Paul F. Baum, Ludwik Dąbrowski ja Tomasz Maszczyk


Jiří Matoušek
Using the Borsuk-Ulam Theorem
Lectures on Topological Methods in Combinatorics and Geometry

The Borsuk-Ulam Theorem

Theorem (Borsuk-Ulam)

Let n be a positive natural number. If $f: S^n \to \mathbb{R}^n$ is continuous, then there exists a pair $(p, -p)$ of antipodal points on S^n such that $f(p) = f(-p)$.

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The logical negation of the theorem

There exists a continuous map $f\colon S^n \to \mathbb{R}^n$ such that for all pairs $(p, -p)$ of antipodal points on S^n we have $f(p) \neq f(-p)$.

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Theorem (equivariant formulation)

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Theorem (Gelfand-Naimark I)

Every commutative C-algebra is naturally isomorphic to the algebra of all continuous complex-valued vanishing-at-infinity functions on a locally compact Hausdorff space.*

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Copernican-style revolution

Given a compact Hausdorff space of points, we can define the C*-algebra of functions on the space, but the central concept is that of a commutative C*-algebras, and points appear as characters (algebra homomorphisms into $\mathbb{C}$) rather than as primary objects. We think of noncommutative unital C*-algebras as algebras of functions on compact quantum spaces.

Banach-Simons Semester

1 Sep – 30 Nov 2016, Simons Semester in the Banach Center

NONCOMMUTATIVE GEOMETRY THE NEXT GENERATION

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Noncommutative Geometry the Next Generation

4–17 September, Będlewo & Warsaw, Master Class on:

Noncommutative geometry and quantum groups

1. Cyclic homology
   by Masoud Khalkhali and Ryszard Nest
2. Noncommutative index theory
   by Nigel Higson and Erik Van Erp
3. Topological quantum groups and Hopf algebras
   by Alfons Van Daele and Stanisław L. Woronowicz
4. Structure and classification of C*-algebras
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19 September – 14 October, 20-hour lecture courses:

1. An invitation to C*-algebras by Karen R. Strung
2. An invitation to Hopf algebras by Réamonn Ó Buachalla
3. Noncommutative topology for beginners by Tatiana Shulman

Conferences

1. 17–21 Oct. Cyclic homology
   J. Cuntz, P. M. Hajac, T. Maszczyk, R. Nest

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4. 21–25 Nov. Structure and classification of C*-algebras
   G. Elliott, K. R. Strung, W. Winter, J. Zacharias

18–22 July 2016, the Fields Institute

GEOMETRY, REPRESENTATION THEORY AND THE BAUM-CONNES CONJECTURE

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Sponsored by:

- The Fields Institute, University of Toronto, Canada
- National Science Foundation, USA
- The Pennsylvania State University, USA

FIELDS

What is a compact quantum group?

Definition (S. L. Woronowicz)

A compact quantum group is a unital C^*-algebra H with a given unital $*$-homorphism $\Delta\colon H \longrightarrow H \otimes_{\min} H$ such that the diagram

$$\begin{array}{ccc}
H & \xrightarrow{\Delta} & H \underset{\min}{\otimes} H \\
\Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \Delta\otimes\mathrm{id}} \\
H \underset{\min}{\otimes} H & \xrightarrow[\mathrm{id}\otimes\Delta]{} & H \underset{\min}{\otimes} H \underset{\min}{\otimes} H
\end{array}$$

commutes and the two-sided cancellation property holds:

$$\{(a\otimes 1)\Delta(b) \mid a, b \in H\}^{\mathrm{cls}} = H \underset{\min}{\otimes} H = \{\Delta(a)(1\otimes b) \mid a, b \in H\}^{\mathrm{cls}}.$$

Here "cls" stands for "closed linear span".

Free actions of compact quantum groups

Let A be a unital C^*-algebra and $\delta : A \to A \otimes_{\min} H$ a unital $*$-homomorphism. We call δ a coaction of H on A (or an action of the compact quantum group (H, Δ) on A) iff

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \otimes_{\min} H$ (counitality)
3. $\ker \delta = 0$ (injectivity).

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3. $\ker \delta = 0$ (injectivity).

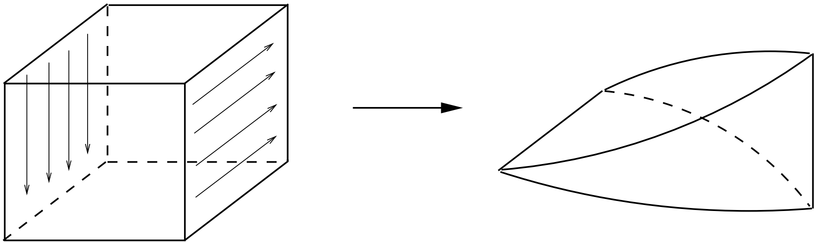
Definition (D. A. Ellwood)

A coaction δ is called free iff

$$\boxed{\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H}\,.$$

Equivariant join construction

For any topological spaces X and Y, one defines the join space $X * Y$ as the quotient of $[0,1] \times X \times Y$ by a certain equivalence relation:



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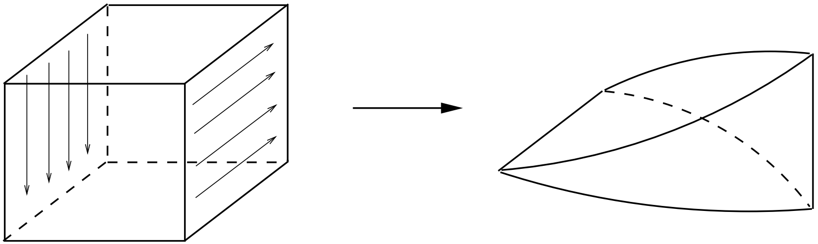

If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G, then the diagonal action of G on the join $X * G$ is again continuous and free.

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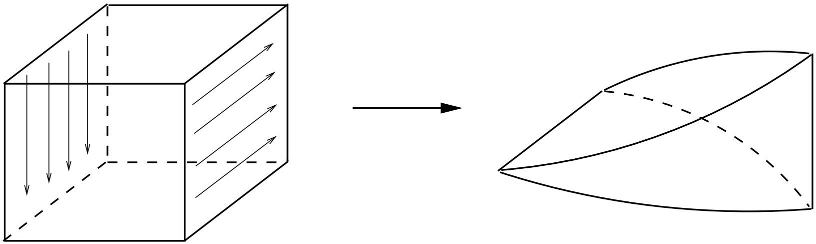

If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G, then the diagonal action of G on the join $X * G$ is again continuous and free. In particular, for the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on S^{n-1}, we obtain a $\mathbb{Z}/2\mathbb{Z}$-equivariant identification $S^n \cong S^{n-1} * \mathbb{Z}/2\mathbb{Z}$ for the antipodal and diagonal actions respectively.

Join formulation and classical generalization

Thus the Borsuk-Ulam Theorem is equivalent to:

Theorem (join formulation)

Let n be a positive natural number. There does not exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^{n-1} * \mathbb{Z}/2\mathbb{Z} \to S^{n-1}$.

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This naturally leads to:

A classical Borsuk-Ulam-type conjecture

Let X be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group G. Then, for the diagonal action of G on $X * G$, there does not exist a G-equivariant continuous map $f : X * G \to X$.

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Corollary

Ageev's conjecture about the Menger compacta.

Equivariant noncommutative join construction

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group (H, Δ) acting freely on a unital C*-algebra A, we define its equivariant join with H to be the unital C*-algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0,1], A) \underset{\min}{\otimes} H \;\middle|\; f(0) \in \mathbb{C} \otimes H,\ f(1) \in \delta(A) \right\}.$$

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Theorem (P. F. Baum, K. De Commer, P. M. H.)

*The *-homomorphism*

$$\mathrm{id} \otimes \Delta \colon\ C([0,1], A) \underset{\min}{\otimes} H \longrightarrow C([0,1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

defines a free action of the compact quantum group (H, Δ) on the equivariant join C-algebra $A \circledast^{\delta} H$.*

Noncommutative Borsuk-Ulam-type conjectures

Conjecture 1

Let A be a unital (nuclear) C*-algebra with a free action of a non-trivial compact quantum group (H, Δ). Then there does not exist an H-equivariant *-homomorphism $A \to A \circledast^{\delta} H$.

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Let A be a unital (nuclear) C*-algebra with a free action of a non-trivial compact quantum group (H, Δ). If A admits a character, then there does not exist an H-equivariant *-homomorphism $H \to A \circledast^{\delta} H$.

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The classical cases

If X is a compact Hausdorff principal G-bundle, $A = C(X)$ and $H = C(G)$, then Conjecture 2 states that the principal G-bundle $X * G$ is not trivializable unless G is trivial. This is clearly true because otherwise $G * G$ would be trivializable, which is tantamount to G being contractible, and the only contractible compact Hausdorff group is trivial.

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Iterated joins of the quantum $SU(2)$ group

Consider the fibration defining the quaternionic projective space:

$$SU(2) * \cdots * SU(2) \cong S^{4n+3}, \quad S^{4n+3}/SU(2) = \mathbb{H}P^n.$$

To obtain a q-deformation of this fibration, we take $H := C(SU_q(2))$ and $A := C(S_q^{4n+3})$ equal to the n-times iterated equivariant join of H. The quantum principal $SU_q(2)$-bundle thus given is *not* trivializable:

Theorem (main)

*There does not exist a $C(SU_q(2))$-equivariant *-homomorphism $f\colon C(SU_q(2)) \longrightarrow C(S_q^{4n+3}) \circledast^{\delta} C(SU_q(2))$.*

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Proof outline: If f existed, there would be an equivariant map F $C(SU_q(2)) \to C(S_q^{4n+3}) \circledast^{\delta} C(SU_q(2)) \to C(SU_q(2)) \circledast^{\Delta} C(SU_q(2))$. Furthermore, for any finite-dimensional representation V of a compact quantum group (H, Δ), the associated finitely-generated projective module $(H \circledast^{\Delta} H)\Box_H V$ is represented by a Milnor idempotent $p_{U^{-1}}$, where U is a matrix of the representation V. If $H := C(SU_q(2))$ and V is the fundamental representation of $SU_q(2)$, then $(H \circledast^{\Delta} H)\Box_H V$ is not stably free by an index paring calculation. This contradicts the existence of F. $\Box$

Quantum Dynamics, 2016–2019

Research and Innovation Staff Exchange network of:
IMPAN (Poland), University of Warsaw (Poland), University of Łódź (Poland), University of Glasgow (G. Britain), University of Aberdeen (G. Britain), University of Copenhagen (Denmark), University of Münster (Germany), Free University of Brussels (Belgium), SISSA (Italy), Penn State University (USA), University of Colorado at Boulder (USA), University of Kansas at Lawrence (USA), University of California at Berkeley (USA), University of Denver (USA), Fields Institute (Canada), University of New Brunswick at Fredericton (Canada), University of Wollongong (Australia), Australian National University (Australia), University of Otago (New Zealand), University Michoacana de San Nicolás de Hidalgo (Mexico).

European Commission

HORIZON 2020

The EU Framework Programme for Research and Innovation