UNIVERSITY OF OULU
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FROM THE NON-CONTRACTIBILITY OF COMPACT QUANTUM GROUPS TO A NONCOMMUTATIVE BORSUK-ULAM-TYPE CONJECTURE

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Joint work with P. F. Baum, L. Dąbrowski and S. Neshveyev.

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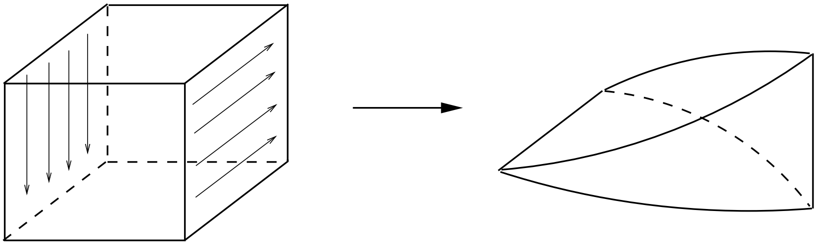
Equivariant join construction

For any topological spaces X and Y, one defines the join space $X * Y$ as the quotient of $[0,1] \times X \times Y$ by a certain equivalence relation:



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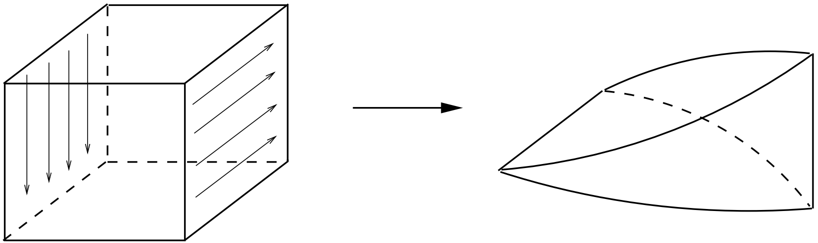

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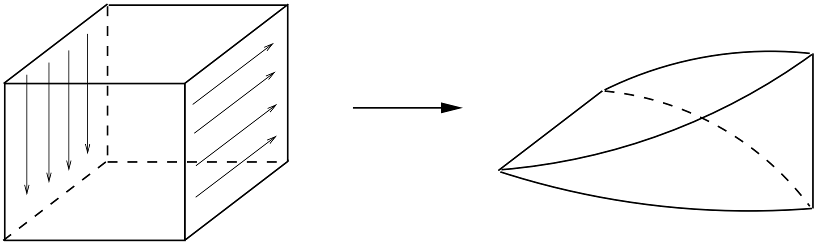

If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G, then the diagonal action of G on the join $X * G$ is again continuous (not obvious!) and free (clear). In particular, for the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on S^{n-1}, we obtain a $\mathbb{Z}/2\mathbb{Z}$-equivariant identification $S^n \cong S^{n-1} * \mathbb{Z}/2\mathbb{Z}$ for the antipodal and diagonal actions respectively.

Classical contractibility

A topological space X is contractible iff there exists a retraction of its inclusion into its cone:

$$\exists\, \varphi\colon CX \to X \quad \text{s.t.} \quad X \hookrightarrow CX \overset{\varphi}{\to} X \quad \text{is} \quad \mathrm{id}_X\,.$$

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If we replace X by a topological group G, then the contractibility of G can be phrased equivariantly:

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The only contractible compact group is the trivial one. Hence, if G is a non-trivial compact group, then γ does not exist. How about $G^{*\,n+1} \longrightarrow G^{*\,n}$?

Join formulation and classical generalization

Theorem (Borsuk-Ulam)

Let n be a positive natural number. There does not exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $S^{n-1} * \mathbb{Z}/2\mathbb{Z} \to S^{n-1}$.

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A classical Borsuk-Ulam-type conjecture

Let X be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group G. Then, for the diagonal action of G on $X * G$, there does not exist a G-equivariant continuous map $f : X * G \to X$.

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Note that the existence of f is equivalent to the existence of $\varphi : CX \to X$ s.t. $X \hookrightarrow CX \xrightarrow{\varphi} X$ is G-equivariant. At the moment, the conjecture is known to hold under the assumption of local triviality. In its full generality, it is deeply related to the celebrated Hilbert-Smith conjecture.

What is a compact quantum group?

Definition (S. L. Woronowicz)

A compact quantum group is a unital C^*-algebra H with a given unital $*$-homorphism $\Delta\colon H \longrightarrow H \otimes_{\min} H$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\quad\Delta\quad} & H \underset{\min}{\otimes} H \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \Delta\otimes\mathrm{id}} \\ H \underset{\min}{\otimes} H & \xrightarrow[\mathrm{id}\otimes\Delta]{} & H \underset{\min}{\otimes} H \underset{\min}{\otimes} H \end{array}$$

commutes and the two-sided cancellation property holds:

$$\{(a\otimes 1)\Delta(b) \mid a, b \in H\}^{\mathrm{cls}} = H \underset{\min}{\otimes} H = \{\Delta(a)(1\otimes b) \mid a, b \in H\}^{\mathrm{cls}}.$$

Here “cls” stands for “closed linear span”.

Free actions of compact quantum groups

Let A be a unital C^*-algebra and $\delta : A \to A \otimes_{\min} H$ a unital $*$-homomorphism. We call δ a coaction of H on A (or an action of the compact quantum group (H, Δ) on A) iff

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\ker \delta = 0$ (injectivity).

Note that the injectivity condition implies

- $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \otimes_{\min} H$ (counitality).

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Definition (D. A. Ellwood)

A coaction δ is called free iff

$$\boxed{\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H}\,.$$

The contractibility of compact quantum spaces

Let A be a unital C*-algebra. We call A unitally contractible iff there exists a *-homomorphism

$$\phi : A \to \mathcal{C}A := \{x \in C([0,1], A) \mid \mathrm{ev}_0(x) \in \mathbb{C}\} \text{ s.t. } \mathrm{ev}_1 \circ \phi = \mathrm{id}_A \,.$$

If A is unitally contractible, then it admits a character and its K-theory coincides with the K-theory of $\mathbb{C}$.

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If (H, Δ) is a compact quantum group, then its contractibility can be phrased equivariantly:

$$\exists\, \phi \colon H \to \mathcal{C}H \quad \text{s.t.} \quad H \overset{\phi}{\to} \mathcal{C}H \overset{\mathrm{ev}_1}{\twoheadrightarrow} H \quad \text{is } H\text{-equivariant.}$$

Indeed, the equivalence follows from the fact that all equivariant morphisms from a locally compact quantum group to itself are *-isomorphisms (R. Meyer, S. Roy, S. L. Woronowicz).

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It follows from the classification of finite-dimensional C*-algebras that the only contractible finite quantum group is the trivial one.

Equivariant noncommutative join construction

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group (H, Δ) acting freely on a unital C*-algebra A, we define its equivariant join with H to be the unital C*-algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0,1], A) \underset{\min}{\otimes} H \;\middle|\; f(0) \in \mathbb{C} \otimes H,\ f(1) \in \delta(A) \right\}.$$

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Theorem (P. F. Baum, K. De Commer, P. M. H.)

*The *-homomorphism*

$$\mathrm{id} \otimes \Delta\colon\; C([0,1], A) \underset{\min}{\otimes} H \longrightarrow C([0,1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

defines a free action of the compact quantum group (H, Δ) on the equivariant join C-algebra $A \circledast^{\delta} H$.*

Noncommutative Borsuk-Ulam-type conjecture

Conjecture

Let A be a unital C*-algebra with a free action of a non-trivial compact quantum group (H, Δ). Then there does not exist an H-equivariant *-homomorphism $A \to A \circledast^{\delta} H$.

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Theorem (torsion case)

Let A be a unital C-algebra with a free action $\delta : A \to A \otimes_{\min} H$ of a non-trivial compact quantum group (H, Δ), and let $A \circledast^{\delta} H$ be the equivariant noncommutative join C*-algebra of A and H with the induced free action of (H, Δ). Then, if H admits a character different from the counit whose finite convolution power is the counit, the following statements are true and equivalent:*

1. $\nexists$ *an H-equivariant $*$-homomorphism $A \longrightarrow A \circledast^{\delta} H$.*
2. $\nexists$ *a $*$-homomorphism $\phi : A \longrightarrow \mathcal{C}A$ such that $\mathrm{ev}_1 \circ \gamma$ is H-equivariant.*

Remarks instead of a proof

The first statement can be reduced to its special case $H = C(\mathbb{Z}/k\mathbb{Z})$, $k > 1$, proven by B. Passer. The second statement follows from the first one for any compact quantum group (H, Δ), and the reverse implication is true when (H, Δ) admits the counit.

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A stronger version of the first statement holds if the compact quantum group (H, Δ) admits a finite-dimensional representation whose K_1-class is not trivial. Hence it holds for the abelian compact quantum groups $C_r^* F_n$, which do not admit the counit. (Here F_n is the free group on n generators.)

Aplications

Since the Toeplitz C*-algebra $\mathcal{T}$ admits a free $\mathbb{Z}/2\mathbb{Z}$-action, it follows from the above theorem that $\mathcal{T}$ is not unitally contractible despite having the same K-theory as $\mathbb{C}$, and despite being a deformation of a contractible space (disc) C*-algebra.

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From the special case $(A, \delta) = (H, \Delta)$ we can conclude that any compact quantum group with a torsion character is not contractible. Therefore, among non-contractible compact quantum groups are compact quantum groups that are: with classical torsion, without a character, with non-trivial K-theory (finite and $C_r^* F_n$ included).