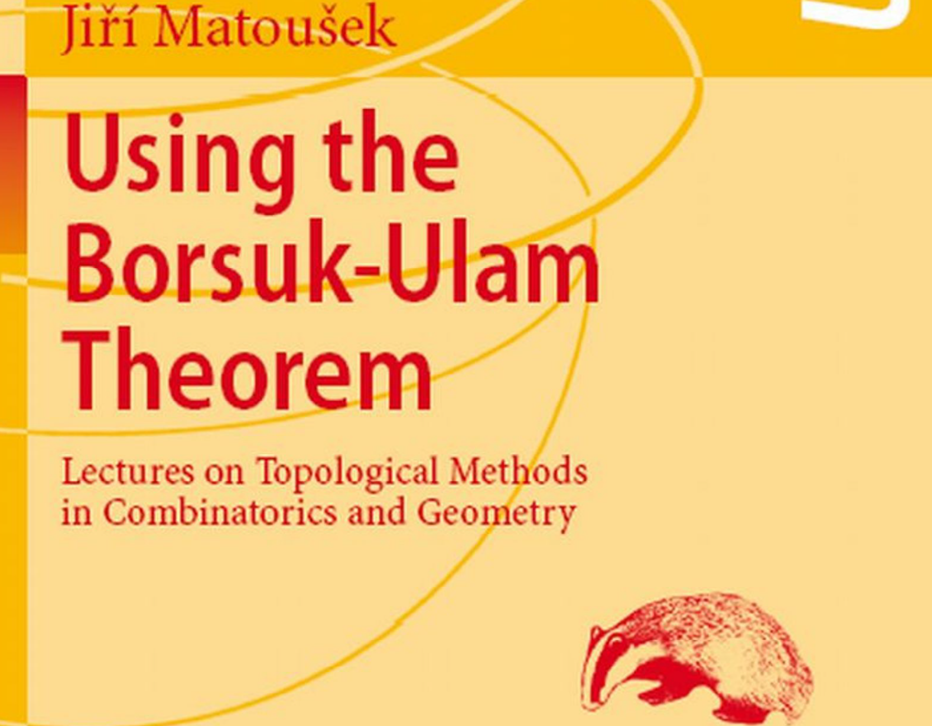
THERE AND BACK AGAIN: FROM THE BORSUK-ULAM THEOREM TO QUANTUM SPACES

Piotr M. Hajac (IMPAN/Altoona)

Joint work with
P. F. Baum, L. Dąbrowski, S. Neshveyev and T. Maszczyk.

William Penn (1644 – 1718)

As one of the earlier supporters of colonial unification, Penn wrote and urged for a union of all the English colonies in what was to become the United States of America. The democratic principles that he set forth in the Pennsylvania Frame of Government served as an inspiration for the United States Constitution. He developed a forward-looking project for a United States of Europe through the creation of a European Assembly made of deputies that could discuss and adjudicate controversies peacefully. He is therefore considered the very first thinker to suggest the creation of a European Parliament.


Jiří Matoušek
Using the Borsuk-Ulam Theorem
Lectures on Topological Methods in Combinatorics and Geometry

The Borsuk-Ulam Theorem

Theorem (Borsuk-Ulam)

Let n *be a positive natural number. If* $f\colon S^n \to \mathbb{R}^n$ *is continuous, then there exists a pair* $(p, -p)$ *of antipodal points on* S^n *such that* $f(p) = f(-p)$.

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The logical negation of the theorem

There exists a continuous map $f\colon S^n \to \mathbb{R}^n$ such that for all pairs $(p, -p)$ of antipodal points on S^n we have $f(p) \neq f(-p)$.

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Equivalent negation

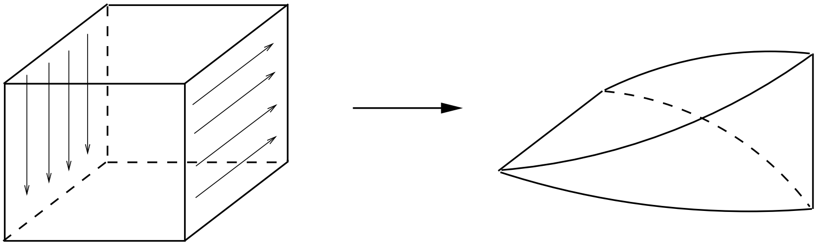
There exists a $\mathbb{Z}/2\mathbb{Z}$-equivalent continuous map $\widetilde{f}\colon S^n \to S^{n-1}$.

Theorem (equivariant formulation)

Let n be a positive natural number. There does not exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^n \to S^{n-1}$.

Equivariant join construction

For any topological spaces X and Y, one defines the join space $X * Y$ as the quotient of $[0,1] \times X \times Y$ by a certain equivalence relation:



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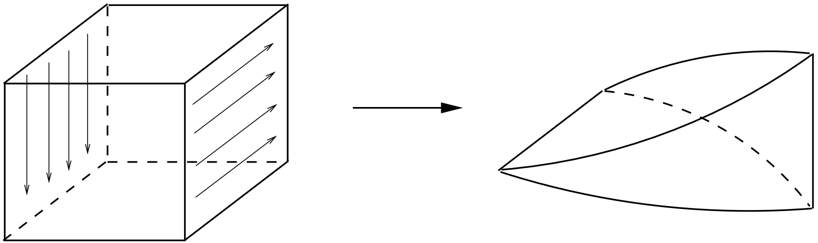

If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G, then the diagonal action of G on the join $X * G$ is again continuous and free.

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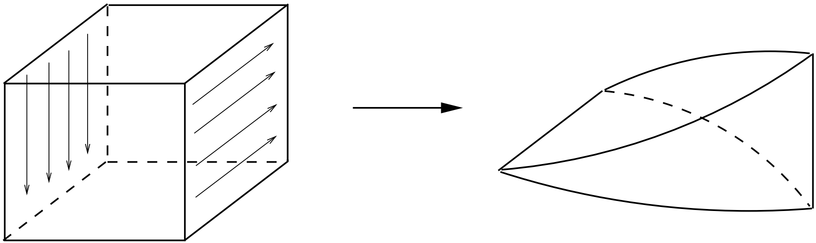

If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G, then the diagonal action of G on the join $X * G$ is again continuous and free. In particular, for the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on S^{n-1}, we obtain a $\mathbb{Z}/2\mathbb{Z}$-equivariant identification $S^n \cong S^{n-1} * \mathbb{Z}/2\mathbb{Z}$ for the antipodal and diagonal actions respectively.

Join formulation and classical generalization

Thus the Borsuk-Ulam Theorem is equivalent to:

Theorem (join formulation)

Let n be a positive natural number. There does not exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^{n-1} * \mathbb{Z}/2\mathbb{Z} \to S^{n-1}$.

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This naturally leads to:

A classical Borsuk-Ulam-type conjecture

Let X be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group G. Then, for the diagonal action of G on $X * G$, there does not exist a G-equivariant continuous map $f : X * G \to X$.

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At the moment, the conjecture is known to hold under the assumption of local triviality. Without this assumption, it implies a version of the Hilbert-Smith conjecture.

Game of Theorems

"When you play the game of theorems, you prove or you fail. There is no middle ground."

Cersei Lannister to Eddard Stark

Quantum Dynamics, 2016–2019

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Every commutative C-algebra is naturally isomorphic to the algebra of all continuous complex-valued vanishing-at-infinity functions on a locally compact Hausdorff space.*

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Copernican-style revolution

Given a compact Hausdorff space of points, we can define the C*-algebra of functions on the space, but the central concept is that of a commutative C*-algebras, and points appear as characters (algebra homomorphisms into $\mathbb{C}$) rather than as primary objects. We think of noncommutative unital C*-algebras as algebras of functions on compact quantum spaces.

What is a compact quantum group?

Definition (S. L. Woronowicz)

A compact quantum group is a unital C^*-algebra H with a given unital $*$-homorphism $\Delta\colon H \longrightarrow H \otimes_{\min} H$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \underset{\min}{\otimes} H \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \Delta\otimes\mathsf{id}} \\ H \underset{\min}{\otimes} H & \xrightarrow[\mathsf{id}\otimes\Delta]{} & H \underset{\min}{\otimes} H \underset{\min}{\otimes} H \end{array}$$

commutes and the two-sided cancellation property holds:

$$\{(a\otimes 1)\Delta(b) \mid a, b \in H\}^{\text{cls}} = H \underset{\min}{\otimes} H = \{\Delta(a)(1\otimes b) \mid a, b \in H\}^{\text{cls}}.$$

Here "cls" stands for "closed linear span".

Free actions of compact quantum groups

Let A be a unital C^*-algebra and $\delta : A \to A \otimes_{\min} H$ a unital $*$-homomorphism. We call δ a coaction (or an action of the compact quantum group (H, Δ) on A) iff

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \otimes_{\min} H$ (counitality),
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Definition (D. A. Ellwood)

A coaction δ is called free iff

$$\boxed{\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H}\,.$$

Equivariant noncommutative join construction

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group (H, Δ) acting freely on a unital C*-algebra A, we define its equivariant join with H to be the unital C*-algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0,1], A) \underset{\min}{\otimes} H \;\middle|\; f(0) \in \mathbb{C} \otimes H,\; f(1) \in \delta(A) \right\}.$$

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Theorem (P. F. Baum, K. De Commer, P. M. H.)

*The *-homomorphism*

$$\mathrm{id} \otimes \Delta\colon\ C([0,1], A) \underset{\min}{\otimes} H \longrightarrow C([0,1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

defines a free action of the compact quantum group (H, Δ) on the equivariant join C-algebra $A \circledast^{\delta} H$.*

Noncommutative Borsuk-Ulam-type conjectures

Conjecture 1

Let A be a unital C*-algebra with a free action of a non-trivial compact quantum group (H, Δ). Then there does not exist an H-equivariant *-homomorphism $A \to A \circledast^{\delta} H$. (Known to hold for (H, Δ) with classical torsion.)

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Conjecture 2

Let A be a unital C*-algebra with a free action of a non-trivial compact quantum group (H,Δ). If A admits a character, then there does not exist an H-equivariant *-homomorphism $H \to A \circledast^{\delta} H$. (Follows from Conjecture 1.)

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Classical cases

If X is a compact Hausdorff principal G-bundle, $A = C(X)$ and $H = C(G)$, then Conjecture 2 states that the principal G-bundle $X * G$ is not trivializable unless G is trivial. This is clearly true because otherwise $G * G$ would be trivializable, which is tantamount to G being contractible, and the only contractible compact Hausdorff group is the trivial one.

Iterated joins of the quantum $SU(2)$ group

Consider the fibration defining the quaternionic projective space:

$$SU(2) * \cdots * SU(2) \cong S^{4n+3}, \quad S^{4n+3}/SU(2) = \mathbb{H}P^n.$$

To obtain a q-deformation of this fibration, we take $H := C(SU_q(2))$ and $A := C(S_q^{4n+3})$ equal to the n-times iterated equivariant join of H. The thus given quantum principal $SU_q(2)$-bundle is *not* trivializable:

Theorem

*There does not exist a $C(SU_q(2))$-equivariant *-homomorphism*

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$$f\colon C(SU_q(2)) \longrightarrow C(S_q^{4n+3}) \circledast^{\delta} C(SU_q(2)).$$

This theorem holds because $SU_q(2)$ has classical torsion elements. It also follows from the stable non-triviality of the dual tautological quaternionic line bundle:

The tautological quaternionic line bundle

If f existed, there would exist an equivariant map F

$$C(SU_q(2)) \to C(S_q^{4n+3}) \circledast^{\delta} C(SU_q(2)) \to C(SU_q(2)) \circledast^{\Delta} C(SU_q(2)).$$

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Furthermore, for any finite-dimensional representation V of a compact quantum group (H, Δ), the associated finitely generated projective module $(H \circledast^{\Delta} H) \Box_H V$ is represented by a Milnor idempotent $p_{U^{-1}}$, where U is a matrix of the representation V, and an even index pairing calculation for $p_{U^{-1}}$ might be replaced by an odd index pairing calculation for U.

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Now, for $H := C(SU_q(2))$ and V the fundamental representation of $SU_q(2)$, the module $(H \circledast^{\Delta} H) \Box_H V$ is the section module of the dual tautological quaternionic line bundle. It is *not* stably free by the non-vanishing of an index paring of the fundamental representation of $SU_q(2)$ with an appropriate odd Fredholm module. This contradicts the existence of F.