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RANK-TWO MILNOR IDEMPOTENTS FOR THE MULTIPULLBACK QUANTUM COMPLEX PROJECTIVE PLANE

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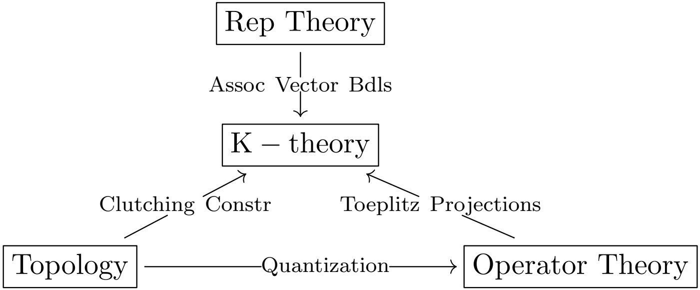
Joint work with
Carla Farsi, Tomasz Maszczyk and Bartosz Zieliński.

Mount Elbert (4400 m)

Road map


Rep Theory
Assoc Vector Bdls
K – theory
Clutching Constr
Toeplitz Projections
Topology
Quantization
Operator Theory

Odd-to-even connecting homomorphism

For any one-surjective pullback diagram of rings

$$\begin{array}{ccccc} R_1 & \longleftarrow & R & \longrightarrow & R_2 \ , \\ & \overset{\pi^1}{\searrow} & & \overset{\pi^2}{\swarrow} & \\ & & R_{12} & & \end{array}$$

there exists the following long exact sequence in algebraic K-theory:

$$\cdots \longrightarrow K_1^{\mathrm{alg}}(R_{12}) \xrightarrow{\partial_{10}^{\mathrm{alg}}} K_0(R) \longrightarrow K_0(R_1 \oplus R_2) \longrightarrow K_0(R_{12}),$$

with $\partial_{10}^{\mathrm{alg}}$ determined by $GL_\infty(R_{12}) \ni U \longmapsto M \in Proj(R)$,

$$\begin{array}{ccccccc} R_1^n & \longleftarrow & & M & & \longrightarrow & R_2^n \\ & \underset{(\pi^1,\cdots,\pi^1)}{\searrow} & & & & \underset{(\pi^2,\cdots,\pi^2)}{\swarrow} & \\ & & R_{12}^n & \xrightarrow{U} & R_{12}^n \ . & & \end{array}$$

Modules associated to piecewise cleft coactions

Let $\mathcal{H}$ be a Hopf algebra, let

$$\begin{array}{ccccc} & & \mathcal{P} & & \\ & \swarrow & & \searrow & \\ \mathcal{P}_1 & \xrightarrow[\tilde{\pi}_1]{} & \mathcal{P}_{12} & \xleftarrow[\tilde{\pi}_2]{} & \mathcal{P}_2 \end{array}$$

be a one-surjective pullback diagram of $\mathcal{H}$-comodule algebras, and let $\gamma_i : \mathcal{H} \to \mathcal{P}_i$ be cleaving maps.

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Clutching Theorem

If V is a finite-dimensional left $\mathcal{H}$-comodule, then the associated left $\mathcal{P}_{12}^{\mathrm{co}\,\mathcal{H}}$-module $\mathcal{P}\Box V$ is the Milnor module for the automorphism of $\mathcal{P}_{12}^{\mathrm{co}\,\mathcal{H}} \otimes V$ given by

$$b \otimes v \longmapsto b(\tilde{\pi}_1 \circ \gamma_1)(v_{(-2)})(\tilde{\pi}_2 \circ \gamma_2)^{-1}(v_{(-1)}) \otimes v_{(0)}.$$

Quantum balls and spheres

For the Hong-Szymański quantum balls and Vaksman-Soibelman quantum spheres, we just proved:

Theorem (F. D'Andrea, P. M. H., M. Tobolski)

$\forall\, n \in \mathbb{N} \setminus \{0\}\ \exists$ *a* $U(1)$*-equivariant pullback of C*-algebras:*

$$\begin{array}{ccccc} & & C(S_q^{2n+1}) & & \\ & \swarrow & & \searrow & \\ C(S_q^{2n-1}) & & & & C(B_q^{2n}) \otimes C(S^1) \\ & \searrow & & \swarrow & \\ & & C(S_q^{2n-1}) \otimes C(S^1). & & \end{array}$$

Bundles over quantum complex projective spaces

Corollary

$\forall\, n \in \mathbb{N} \setminus \{0\}\ \exists$ *a pullback of C*-algebras:*

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\begin{array}{ccccc}
 & & C(\mathbb{C}P_q^n) & & \\
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$$
\begin{array}{ccccc}
K_0(C(\mathbb{C}P_q^n)) & \longrightarrow & K_0(C(\mathbb{C}P_q^{n-1})) \oplus K_0(C(B_q^{2n})) & \longrightarrow & K_0(C(S_q^{2n-1})) \\
{\color{red}\partial_{10}}\uparrow & & & & \downarrow \partial_{01} \\
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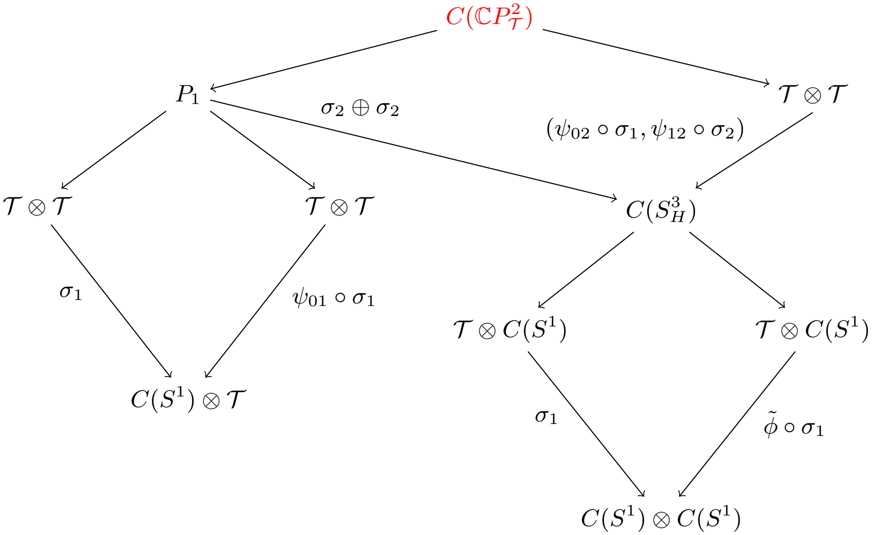
Multipullback quantum complex projective



Multipullback quantum spheres S_H^{2N+1}

$C(S_H^{2N+1})$ is the C*-subalgebra of $\prod_{i=0}^{N} \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i}$ defined by the compatibility conditions prescribed by the following diagrams ($0 \leq i < j \leq N$, $\otimes$-supressed):

$$\mathcal{T}^i C(S^1) \mathcal{T}^{N-i} \xrightarrow{\sigma_j} \mathcal{T}^i C(S^1) \mathcal{T}^{j-i-1} C(S^1) \mathcal{T}^{N-j} \xleftarrow{\sigma_i} \mathcal{T}^j C(S^1) \mathcal{T}^{N-j}.$$

Here $\sigma_k := \mathrm{id}^k \otimes \sigma \otimes \mathrm{id}^{N-k}$ with domains and codomains determined by the context.

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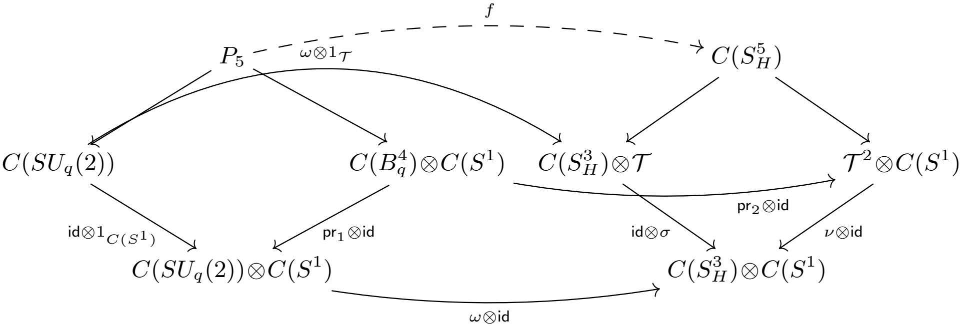
We equip all C*-algebras in the diagrams with the diagonal actions of $U(1)$. Since all morphisms in the diagrams are $U(1)$-equivariant, we obtain the diagonal $U(1)$-action on $C(S_H^{2N+1})$.

Reducing to the quantum-group case



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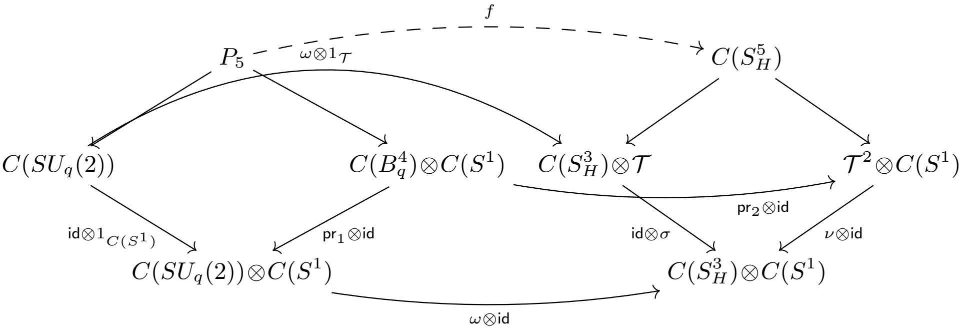

K-Isomorphism Lemma

The above *-homomorphisms are $U(1)$-equivariant, and the induced *-homomorphisms on fixed-point subalgebras yield isomorphisms on K-groups.

Main result

Definition

Let $k \in \mathbb{Z}$. We call the left $C(\mathbb{C}P^2_{\mathcal{T}})$-module

$$L_k := \{a \in C(S^5_H) \mid \forall\, \lambda \in U(1) : \alpha_\lambda(a) = \lambda^k a\}$$

the section module of the associated line bundle of winding number k.

Theorem

The group $K_0(C(\mathbb{C}P^2_{\mathcal{T}}))$ is freely generated by elements

$$[1], \quad [L_1] - [1], \quad [L_1 \oplus L_{-1}] - [2].$$

Furthermore, $L_1 \oplus L_{-1} \cong C(\mathbb{C}P^2_{\mathcal{T}}) \oplus C(\mathbb{C}P^2_{\mathcal{T}})e$. Here $e \in C(\mathbb{C}P^2_{\mathcal{T}})$ is an idempotent such that $C(\mathbb{C}P^2_{\mathcal{T}})e$ cannot be realized as a finitely generated projective module associated with the $U(1)$-C-algebra $C(S^5_H)$ of Heegaard quantum 5-sphere.*

Proof outline

1. Take the $SU_q(2)$-prolongation $P_5\square^{\mathcal{O}(U(1))}\mathcal{O}(SU_q(2))$. Then take the fundamental representation $\mathbb{C}^2$ of $SU_q(2)$, and compute the clutching matrix of the associated module

$$P_5\square^{\mathcal{O}(U(1))}\mathcal{O}(SU_q(2))\square^{\mathcal{O}((SU_q(2))}\mathbb{C}^2 = L'_1 \oplus L'_{-1}$$

from the Clutching Theorem.

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2. The clutching matrix turns out to be the fundamental representation matrix U_f, so its class generates $K_1(C(SU_q(2)))$. Now it follows from the six-term Mayer-Vietoris exact sequence that the Milnor class

$$\partial_{10}([U_f]) = [L'_1 \oplus L'_{-1}] - 2$$

is the third generator of $K_0(P_5^{U(1)})$.

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3. Finally, combining the above with the Isomorphism Lemma and the Pulling-back Corollary, we conclude that $[L_1 \oplus L_{-1}] - 2$ is the third generator of $K_0(C(\mathbb{C}P^2_{\mathcal{T}}))$.

Tentative plan of conferences

New Geometry of Quantum Dynamics planned conferences:

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- The Fields Institute, Toronto, 22 July – 16 August 2019 (pending approval)