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Overview

Partial representations of groups

- What is a partial action / partial representation of a group G?
- How do partial representations form a monoidal category?

$\rightsquigarrow$ the transformation groupoid $B^e_G \rtimes G$ for the partial Bernoulli shift

Partial representations of Hopf algebras

- What is a partial representation of a Hopf algebra H?
- What is the analogue of the partial Bernoulli shift?

$\rightsquigarrow$ a Hopf algebroid counterpart H_{par} to the transformation groupoid

The Bernoulli shift of a discrete quantum group

- Conceptual explanation of the construction of H_{par} on the algebraic level and on the level of C^*-algebras.

Partial actions of discrete groups

Partial actions of discrete groups

“A partial action is the restriction of an action to a non-invariant subset.”

Definition A *partial action* of a discrete group G *on a set* X is given by

- a subset $D_g \subseteq X$ for each $g \in G$
- a bijection $\theta_g \colon D_{g^{-1}} \to D_g$ for each $g \in G$

such that

- $D_e = X$ and $\theta_e = \mathrm{id}_X$,
- $\theta_g \circ \theta_h \subseteq \theta_{gh}$ for all $g, h \in G$ as partial maps.

Example The restriction of an action $G \circlearrowright Y$ to a subset $X \subseteq Y$:

- $D_{g^{-1}} := \{x \in X : g(x) \in X\} = X \cap g^{-1}(X)$ for each $g \in G$
- $\theta_g(x) = g(x)$ for each $g \in G$ and $x \in D_{g^{-1}}$

Variants Partial actions on *spaces*, *algebras*, *vector spaces*, . . .

The partial Bernoulli shift and disconnected partial actions

Example Let G act on $\mathcal{P}(G) \cong \{0,1\}^G$ by left translation. Restriction to

$$B_G^e = \{A \subseteq G : e \in A\} \subset \mathcal{P}(G)$$

yields the *partial Bernoulli shift*, where each $g \in G$ acts as

$$\{A \subseteq G : e, g^{-1} \in A\} = D_{g^{-1}} \xrightarrow{g\cdot -} D_g = \{A \subseteq G : g, e \in A\} .$$

Definition A partial action of G on

- a space X is *disconnected* if each $D_g \subseteq X$ is clopen,
- an algebra A is *disconnected* if each $D_g \subseteq A$ is a direct summand,
- a vector space is *disconnected* if we are given complements $D_g^{\perp}$.

A partial action $G \circlearrowright X$ gives rise to a transformation groupoid $X \rtimes G$.

Proposition (disconnected partial actions of G) $\tilde{\leftrightarrow}$ (actions of $B_G^{\varepsilon} \rtimes G$).

The link between partial actions and groupoid actions

The transformation groupoid $B_G^e \rtimes G$ consists of all arrows of the form

$$gA \xleftarrow{g} A \qquad (g \in G,\ A \subseteq G,\ e, g^{-1} \in A).$$

Proposition (disconnected partial actions of G) $\tilde{\leftrightarrow}$ (actions of $B_G^{\varepsilon} \rtimes G$).

Idea of proof Given a partial action $((D_g)_g, (\theta_g)_g)$ on a space X, define

- $\pi : X \to B_G^e$ by $x \mapsto \{g \in G : x \in D_{g^{-1}}\}$
- $(\ gA \xleftarrow{g} A\ ) \cdot x := \theta_g(x)$, where $A = \pi(x)$ implies $g^{-1} \in A$ and $x \in D_{g^{-1}} = \mathrm{Dom}(\theta_g)$.

Corollary Partial representations of G (:=disconnected partial actions on vector spaces) carry a monoidal product $\boxtimes$, where

$$V \boxtimes W = \bigoplus_{A \in B_G^e} (V_A \otimes W_A), \quad V_A = \left(\textstyle\bigcap_{g \in A} D_g\right) \cap \left(\textstyle\bigcap_{g \notin A} D_g^{\perp}\right),\ W_A = \ldots$$

Partial actions of Hopf algebras

Partial representations of Hopf algebras

Definition A *partial representation* of a *Hopf algebra H* on a vector space V is a linear map $\pi\colon H \to \mathrm{End}(V)$ such that $\pi(1_H) = \mathrm{id}_V$ and

- $\pi(kh_{(1)})\pi(S(h_{(2)})) = \pi(k)\pi(h_{(1)})\pi(S(h_{(2)}))$,
  $\pi(S(k_{(1)}))\pi(k_{(2)}h) = \pi(S(k_{(1)}))\pi(k_{(2)})\pi(h) \qquad (h, k \in H)$;
- as above with S replaced by S^{-1}.

Example (partial rep.s of G) $\overset{\sim}{\leftrightarrow}$ (partial rep.s of the group algebra kG).

Theorem (Alves-Batista-Vercruysse '15)

- There exists an algebra H_{par} whose representations correspond bijectively to partial representations of H.
- $H_{\mathrm{par}} \cong A\#H$ for a subalgebra $A \subset H_{\mathrm{par}}$ with a partial action of H.
- H_{par} is a *Hopf algebroid*.
- Partial representations of H carry a monoidal product.

The partial Bernoulli shift of a discrete quantum group

Partial coactions of C^*-bialgebras

Definition A *partial coaction* of a C^*-bialgebra $(\mathcal{H}, \Delta)$ on a C^*-algebra $\mathcal{A}$ is a $*$-homomorphism $\delta\colon \mathcal{A} \to M(\mathcal{A} \otimes \mathcal{H})$ such that

- $\delta(\mathcal{A})(1 \otimes \mathcal{H}) \subseteq \mathcal{A} \otimes \mathcal{H}$,
- δ extends to a strict $*$-homomorphism from $M(\mathcal{A})$ to $M(\mathcal{A} \otimes \mathcal{H})$,
- $(\delta \otimes \mathrm{id})(\delta(a)) = (\delta(1) \otimes 1)(\mathrm{id} \otimes \Delta)(\delta(a))$ for all $a \in A$.

Example Given a coaction on a C^*-algebra $\mathcal{B}$ and an ideal $\mathcal{A} \subseteq \mathcal{B}$, the coaction restricts to a partial coaction on $\mathcal{A}$.

Kraken, Quast, T. Partial actions of C*-quantum groups I: Restriction and Globalization. *to appear in Banach J. Math. Analysis*. arXiv:1703.06546.

Theorem (Kraken, Quast, T.) Let $(\mathcal{H}, \Delta)$ be a *C^*-quantum group*, where $\mathcal{H}$ is *nuclear*. Then every *regular* partial coaction of $(\mathcal{H}, \Delta)$ is the restriction of a coaction, and there exists a unique *minimal* choice.

Partial coactions of discrete quantum groups

Let $\mathbb{G}$ be a *discrete quantum group*, given by $(C_0(\mathbb{G}),\Delta)$. Then

- $C_0(\mathbb{G}) \cong \oplus_\alpha M_{n_\alpha}(\mathbb{C})$, where the sum is taken over $\alpha \in \mathrm{Irr}(\hat{\mathbb{G}})$
- $\mathcal{O}(\mathbb{G}) := \oplus_\alpha^{\mathrm{alg}} M_n(\mathbb{C})$ is a multiplier Hopf algebra
- $\mathbb{C}\mathbb{G} := \oplus_\alpha^{\mathrm{alg}} \mathrm{Hom}(M_n(\mathbb{C}),\mathbb{C})$ is a Hopf algebra
- $C_0(\mathbb{G})$ has a counit $\varepsilon\colon C_0(\mathbb{G}) \to \mathbb{C}$.

Examples

- If δ is a partial coaction of $(C_0(\mathbb{G}),\Delta)$ on $\mathcal{A}$ s.t. $(\mathrm{id}\otimes\varepsilon)\delta = \mathrm{id}$, then
$$h\cdot a := (\mathrm{id}\otimes h)(\delta(a)) \qquad (a\in\mathcal{A}, h\in\mathbb{C}\mathbb{G}).$$
defines a partial action of $\mathbb{C}\mathbb{G}$ on $\mathcal{A}$.
- If $\mathbb{G} = G$ is a classical discrete group, counital partial coactions of $(C_0(G),\Delta)$ correspond to disconnected partial actions of G.

The Bernoulli shift of a discrete quantum group I

For a group G, the space $\mathcal{P}(G)$ comes with a tautological subset
$$\{(A,g) : g\in A\}\subset \mathcal{P}(G)\times G.$$
For $\mathbb{G}$, we define a quantum space $\mathbb{B}_{\mathbb{G}}$ in terms of a projection
$$p\in M(C(\mathbb{B}_{\mathbb{G}})\otimes C_0(\mathbb{G})).$$

Write $C_0(\mathbb{G})\cong\oplus_\alpha M_{n_\alpha}(\mathbb{C})$ and choose matrix units $e_{ij}^\alpha\in C_0(\mathbb{G})$.

Definition The *quantum Bernoulli space* $\mathbb{B}_{\mathbb{G}}$ of $\mathbb{G}$ is given by the universal unital C^*-algebra $C(\mathbb{B}_{\mathbb{G}})$ with generators p_{ij}^α such that
$$p := \textstyle\sum_{\alpha,i,j} p_{ij}^\alpha\otimes e_{ij}^\alpha$$
is a projection and
$$[p\otimes 1, (\mathrm{id}\otimes\Delta)(p)] = 0.$$

Example If $\mathbb{G} = G$, then $p = \sum p^g\otimes\delta_g$, all p^g commute, and $\mathbb{B}_G\cong\mathcal{P}(G)$.

The Bernoulli shift of a discrete quantum group II

Denote by $p \in M(C(\mathbb{B}_{\mathbb{G}}) \otimes C_0(\mathbb{G}))$ the *universal projection* as before.

Theorem (T.) There exist

- a coaction $\delta: C(\mathbb{B}_{\mathbb{G}}) \to M(C(\mathbb{B}_{\mathbb{G}}) \otimes C_0(\mathbb{G}))$ such that
$$(\delta \otimes \mathrm{id})(p) = (\mathrm{id} \otimes \Delta)(p)$$
- a coaction $\hat{\delta}: C(\mathbb{B}_{\mathbb{G}}) \to M(C(\mathbb{B}_{\mathbb{G}}) \otimes C_0^r(\hat{\mathbb{G}}))$ such that
$$(\hat{\delta} \otimes \mathrm{id})(p) = (\mathrm{id} \otimes \mathrm{ad})(p)$$

and $(C(\mathbb{B}_{\mathbb{G}}), \delta, \hat{\delta})$ is a *braided-commutative Yetter-Drinfeld C^*-algebra*.

Lemma With the notation as above,

- $p_\varepsilon := (\mathrm{id} \otimes \varepsilon)(p) \in C(\mathbb{B}_{\mathbb{G}})$ is a central projection;
- δ restricts to a partial coaction δ_ε on $C(\mathbb{B}_{\mathbb{G}}^\varepsilon) := p_\varepsilon C(\mathbb{B}_{\mathbb{G}})$.

Definition We call $(C(\mathbb{B}_{\mathbb{G}}^\varepsilon), \delta_\varepsilon)$ the *partial Bernoulli shift* of $\mathbb{G}$.

Connection to the construction of Alves-Batista-Vercruysse

- The unital dense $*$-subalgebra $\mathcal{O}(\mathbb{B}_{\mathbb{G}}) \subseteq C(\mathbb{B}_{\mathbb{G}})$ generated by the p_{ij}^α is a braided-commutative Yetter-Drinfeld algebra over $\mathcal{O}(\hat{\mathbb{G}})$.
- $\Rightarrow \mathcal{O}(\mathbb{B}_{\mathbb{G}}) \# \mathcal{O}(\hat{\mathbb{G}})$ is a Hopf algebroid, see:
  T. Brzeziński, G. Militaru. Bialgebroids, $\times_A$-bialgebras and duality. *J. Algebra*, 251(1):279–294, 2002.
- $\mathcal{O}(\mathbb{B}_{\mathbb{G}}) \# \mathcal{O}(\hat{\mathbb{G}})$ is isomorphic to the Hopf algebroid $\mathcal{O}(\hat{\mathbb{G}})_{\mathrm{par}}$:

  Proof:
  - The natural map $\mathcal{O}(\hat{\mathbb{G}}) \to \mathcal{O}(\mathbb{B}_{\mathbb{G}}) \# \mathcal{O}(\hat{\mathbb{G}})$ is a partial representation and extends to a $*$-homomorphism $\Phi: \mathcal{O}(\hat{\mathbb{G}})_{\mathrm{par}} \to \mathcal{O}(\mathbb{B}_{\mathbb{G}}) \# \mathcal{O}(\hat{\mathbb{G}})$.
  - $\mathcal{O}(\hat{\mathbb{G}})_{\mathrm{par}} \cong A \# \mathcal{O}(\hat{\mathbb{G}})$ for an A with a partial action of $\mathcal{O}(\hat{\mathbb{G}})$ which gives an equivariant $\psi: \mathcal{O}(\mathbb{B}_{\mathbb{G}}) \to A$, and $\Psi := \psi \# \mathrm{id}$ is inverse to Φ.

The example $\mathbb{G} = \widehat{S_3}$

Recall: $C(\mathbb{B}_\mathbb{G})$ is generated by the slices of $p \in M(C(\mathbb{B}_\mathbb{G}) \otimes C(\mathbb{G}))$ s.t.

$$[(\mathrm{id} \otimes \Delta)(p), p \otimes 1] = 0. \qquad (\dagger)$$

Now $C(\mathbb{G}) = \mathbb{C}S^3 \xrightarrow[(\varepsilon,\sigma,\rho)]{\cong} \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$, where $(1,2) \equiv \left(1,\ -1,\ \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)\right)$.

Write $p \equiv \left(p^\varepsilon, p^\sigma, \left(\begin{smallmatrix} a & b \\ b^* & d \end{smallmatrix}\right)\right) \in C(\mathbb{B}_\mathbb{G}) \oplus C(\mathbb{B}_\mathbb{G}) \oplus M_2(C(\mathbb{B}_\mathbb{G}))$. Then $(\dagger) \Leftrightarrow$

(1) $(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\dagger)$, which means $[p^\varepsilon \otimes 1, p] = 0$, so that *$p^\varepsilon$ is central*

(2) $(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\dagger)$, which means $[p, p] = 0$

(3) $(\mathrm{id} \otimes \sigma \otimes \rho)(\dagger)$, which means, as $(\sigma \otimes \rho)\Delta = \mathrm{Ad}_R \circ \rho$ with $R = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$,

$$\left[\begin{pmatrix} p^\sigma & \\ & p^\sigma \end{pmatrix}, \left(\begin{smallmatrix} a & -b \\ -b^* & d \end{smallmatrix}\right)\right] = 0,$$ that is, *p^σ is central* as well

(4) $(\mathrm{id} \otimes \rho \otimes \sigma)(\dagger)$ means $\left[\left(\begin{smallmatrix} a & b \\ b^* & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a & -b \\ -b^* & d \end{smallmatrix}\right)\right] = 0$, that is, $ab = bd$

(5) $(\mathrm{id} \otimes \rho \otimes \rho)(\dagger)$ means that, after decomposing $(\rho \otimes \rho)\Delta \sim \varepsilon \oplus \sigma \oplus \rho$,

$$\left[\begin{pmatrix} a & & b & \\ & a & & b \\ b^* & & d & \\ & b^* & & d \end{pmatrix}, \begin{pmatrix} d & & & b^* \\ & p^+ & p^- & \\ & p^- & p^+ & \\ b & & & a \end{pmatrix}\right] = 0,$$ where $p^\pm = \frac{1}{2}(p^\varepsilon \pm p^\sigma)$ is central,

which means $[a, d] = 0,\ b^2 = 0,\ b = ba = db,\ p^- a = p^- d.$

Plans

- Study $C(\mathbb{B}^\varepsilon_\mathbb{G})$, e.g. if $\mathbb{G}$ is the dual of a finite non-abelian group.
- Equip $C(\mathbb{B}^\varepsilon_\mathbb{G}) \rtimes C_0(\mathbb{G})$ with the structure of a C^*-quantum groupoid. For this, we can use the von Neumann algebraic results in:

  Enock and T. Measured quantum transformation groupoids.
  J. Noncommut. Geom., 10(3):1143–1214, 2016.
- Relate partial representations of $\mathbb{G}$ to rep.s of $C(\mathbb{B}^\varepsilon_\mathbb{G}) \rtimes C_0(\mathbb{G})$ and thus obtain a monoidal product on partial representations of $\mathbb{G}$.
- Extend the notion of a partial coaction to cover partial actions of groups on spaces that are *not* disconnected.