A Gromov-Hausdorff Distance for Hilbert Modules

Frédéric Latrémolière

UNIVERSITY of DENVER

Caltech
November 9th, 2016

My program

Geometry of classes of quantum metric spaces

- Study *the geometry of entire classes* of quantum metric spaces using noncommutative analogues of the *Gromov-Hausdorff distance*: the *Gromov-Hausdorff propinquity* family.
- A quantum metric space is a noncommutative generalization of the algebra of Lipschitz functions on a metric space.
- Examples: finite dimensional approximations, perturbations of metrics, compactness theorem, ...

My program

Geometry of classes of quantum metric spaces

- Study *the geometry of entire classes* of quantum metric spaces using noncommutative analogues of the *Gromov-Hausdorff distance*: the *Gromov-Hausdorff propinquity* family.
- A quantum metric space is a noncommutative generalization of the algebra of Lipschitz functions on a metric space.
- Examples: finite dimensional approximations, perturbations of metrics, compactness theorem, ...

Geometry of classes of modules

- Define a *geometry for classes of modules* over quantum metric spaces: the *modular propinquity*.
- Motivated by K-theory, KK-theory, Morita equivalence, as quantum vector bundles, ...
- Examples: Heisenberg modules over quantum 2-tori.

Structure of the talk

1. Compact Quantum Metric Spaces
2. Convergence of quasi-Leibniz quantum compact metric space
3. Metrized quantum vector bundles
4. The modular Propinquity

GPS

1. Compact Quantum Metric Spaces

2. Convergence of quasi-Leibniz quantum compact metric space

3. Metrized quantum vector bundles

4. The modular Propinquity

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\mathsf{L}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)} : x, y \in X, x \neq y \right\}$$

for all $f \in C(X)$ (allowing ∞).

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\mathsf{L}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{d}(x,y)} : x, y \in X, x \neq y \right\}$$

for all $f \in C(X)$ (allowing ∞).

The *Monge-Kantorovich metric* on $\mathscr{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\left\{ \left| \int_X f \, d\mu - \int_X f \, d\nu \right| : f \in \mathfrak{sa}(C(X)), \mathsf{L}(f) \leqslant 1 \right\}.$$

Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; Kerr, 02; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

1. $\mathfrak{A}$ is a *unital C*-algebra*,

Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; Kerr, 02; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

1. $\mathfrak{A}$ is a *unital C*-algebra*,
2. L is a *seminorm* defined on a dense Jordan-Lie subalgebra $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$,
3. $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,

Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; Kerr, 02; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

1. $\mathfrak{A}$ is a *unital C*-algebra*,
2. L is a *seminorm* defined on a dense Jordan-Lie subalgebra $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$,
3. $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,
4. The *Monge-Kantorovich metric* mk_{L}, defined for any $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ by:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\left\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}), \mathsf{L}(a) \leqslant 1\right\}$$

metrizes the *weak* topology on $\mathscr{S}(\mathfrak{A})$*,

Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; Kerr, 02; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

1. $\mathfrak{A}$ is a *unital C*-algebra*,
2. L is a *seminorm* defined on a dense Jordan-Lie subalgebra $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$,
3. $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,
4. The *Monge-Kantorovich metric* mk_{L}, defined for any $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ by:

   $$\mathsf{mk}_{\mathsf{L}}(\varphi,\psi) = \sup\left\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}), \mathsf{L}(a) \leqslant 1\right\}$$

   metrizes the *weak* topology on $\mathscr{S}(\mathfrak{A})$*,
5. $\max\{\mathsf{L}(a \circ b), \mathsf{L}(\{a,b\})\} \leqslant F(\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{B}}, \mathsf{L}(a), \mathsf{L}(b))$,

Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; Kerr, 02; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

1. $\mathfrak{A}$ is a *unital C*-algebra*,
2. L is a *seminorm* defined on a dense Jordan-Lie subalgebra $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$,
3. $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,
4. The *Monge-Kantorovich metric* mk_{L}, defined for any $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ by:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\left\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}), \mathsf{L}(a) \leqslant 1\right\}$$

   metrizes the *weak* topology on $\mathscr{S}(\mathfrak{A})$*,
5. $\max\{\mathsf{L}(a \circ b), \mathsf{L}(\{a, b\})\} \leqslant F(\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{B}}, \mathsf{L}(a), \mathsf{L}(b))$,
6. L is lower semi-continuous wrt $\|\cdot\|_{\mathfrak{A}}$.

Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; Kerr, 02; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

1. $\mathfrak{A}$ is a *unital C*-algebra*,
2. L is a *seminorm* defined on a dense Jordan-Lie subalgebra $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$,
3. $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,
4. The *Monge-Kantorovich metric* mk_{L}, defined for any $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ by:
$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\left\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}), \mathsf{L}(a) \leqslant 1\right\}$$
metrizes the *weak* topology on $\mathscr{S}(\mathfrak{A})$*,
5. $\max\{\mathsf{L}(a \circ b), \mathsf{L}(\{a, b\})\} \leqslant F(\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{B}}, \mathsf{L}(a), \mathsf{L}(b))$,
6. L is lower semi-continuous wrt $\|\cdot\|_{\mathfrak{A}}$.

We call L an *L-seminorm*.

Ergodic actions of compact metric groups

Theorem (Rieffel, 98)

Let G be a *compact group* endowed with a *continuous length function* ℓ. Let α be an *action* of G on some *unital C*-algebra* $\mathfrak{A}$. Set:

$$\forall a \in \mathfrak{A} \quad \mathsf{L}(a) = \sup\left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1_G\} \right\}.$$

$(\mathfrak{A}, \mathsf{L})$ is a *Leibniz quantum compact metric space* if and only if $\left\{a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a\right\} = \mathbb{C}1_{\mathfrak{A}}$.

Ergodic actions of compact metric groups

Theorem (Rieffel, 98)

Let G be a *compact group* endowed with a *continuous length function* ℓ. Let α be an *action* of G on some *unital C*-algebra* $\mathfrak{A}$. Set:

$$\forall a \in \mathfrak{A} \quad \mathsf{L}(a) = \sup\left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1_G\} \right\}.$$

$(\mathfrak{A}, \mathsf{L})$ is a *Leibniz quantum compact metric space* if and only if $\left\{ a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a \right\} = \mathbb{C}1_{\mathfrak{A}}$.

Example: Quantum tori

- $G = \mathbb{T}^d$,
- $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$ (universal for $U_j U_k = \sigma(j,k) U_{j+k}$),
- α : dual action ($\alpha^z U_j = z^j U_j$).
- Associated with a differential calculus when ℓ from invariant Riemannian metric.

Spectral triples and quantum metrics

Theorem (Rieffel, 02; Ozawa-Rieffel, 05; Christ-Rieffel, 15)

Let G be a *discrete group*, l the *word-length function* on G for some finite generator set, and π be the left regular representation of $C^\ast_{\mathrm{red}}(G)$ on $\ell^2(G)$. For all $\xi \in \ell^2(G)$, set $D\xi : g \in G \mapsto \ell(g)\xi(g)$. If G is hyperbolic or nilpotent, then $\left(C^\ast_{\mathrm{red}}(G), |||[D,\pi(\cdot)]|||_{\mathscr{H}}\right)$ is a Leibniz quantum compact metric space.

Spectral triples and quantum metrics

Theorem (Rieffel, 02; Ozawa-Rieffel, 05; Christ-Rieffel, 15)

Let G be a *discrete group*, l the *word-length function* on G for some finite generator set, and π be the left regular representation of $C^*_{\mathrm{red}}(G)$ on $\ell^2(G)$. For all $\xi \in \ell^2(G)$, set $D\xi : g \in G \mapsto \ell(g)\xi(g)$. If G is hyperbolic or nilpotent, then $\left(C^*_{\mathrm{red}}(G), |||[D,\pi(\cdot)]|||_{\mathscr{H}}\right)$ is a Leibniz quantum compact metric space.

Theorem (Aguilar and L., 2015)

Let $\mathfrak{A} = \varinjlim \mathfrak{A}_n$ with $\mathfrak{A}_n$ *f.d.* for all $n \in \mathbb{N}$. if $\mathfrak{A}$ is unital and has a faithful tracial state τ, and if for all $a \in \mathfrak{A}$ we set:

$$\mathsf{L}(a) = \sup\left\{ \frac{\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}}}{\beta(n)} : n \in \mathbb{N} \right\}$$

where $\beta \in (0,\infty)^{\mathbb{N}}$ with $\lim_\infty \beta = 0$, and $\mathbb{E}_n : \mathfrak{A} \twoheadrightarrow \mathfrak{A}_n$ is the conditional expectation with $\tau \circ \mathbb{E}_n = \tau$, then $(\mathfrak{A}, \mathsf{L})$ *is a quasi-Leibniz quantum compact metric space*.

Other Examples

1. Quantum metrics from the standard spectral triple on quantum tori (Rieffel, 98 and 02)
2. Connes-Landi spheres (H. Li, 03)
3. Conformal deformations of quantum metric spaces from spectral triples (L., 15)
4. Curved quantum tori of Dabrowsky and Sitarz (L., 15)
5. Group C*-algebras for groups with rapid decay (Antonescu, Christensen, 2004)
6. Noncommutative Solenoids (L., Packer, 16)
7. Certain C*-crossed-products (J. Bellissard, M. Marcolli, Reihani, 10), (involves my work on locally compact quantum metric space).

Lipschitz morphisms and Quantum Isometries

Theorem-Definition (Lipschitz Morphisms)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *k-Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that any of the following equivalent statement holds:

1. $\varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$ is a *k-Lipschitz map from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ to $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$*,
2. (Rieffel, 00) $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant k \mathsf{L}_{\mathfrak{A}}$,
3. (L., 16) $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

Lipschitz morphisms and Quantum Isometries

Theorem-Definition (Lipschitz Morphisms)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *k-Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that any of the following equivalent statement holds:

1. $\varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$ is a *k-Lipschitz map from* $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ *to* $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$,
2. (Rieffel, 00) $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant k \mathsf{L}_{\mathfrak{A}}$,
3. (L., 16) $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

Theorem (McShane, 1934)

Let (Z, m) be a metric space and $X \subseteq Z$ not empty. If $f : X \to \mathbb{R}$ is a k-Lipschitz function on X then there exists a k-Lipschitz function $g : Z \to \mathbb{R}$ such that g restricts to f on X.

Lipschitz morphisms and Quantum Isometries

Theorem-Definition (Lipschitz Morphisms)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *k-Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that any of the following equivalent statement holds:

1. $\varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$ is a *k-Lipschitz map from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ to $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$*,
2. (Rieffel, 00) $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant k \mathsf{L}_{\mathfrak{A}}$,
3. (L., 16) $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

Definition (Rieffel (98), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that:

$$\mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.$$

A *full quantum isometry* π is a *-isomorphism such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

The Gromov-Hausdorff Distance

Definition

For any two compact metric spaces (X, m_X) and (Y, m_Y), we define $\mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y)$ as:

$$\left\{ (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \middle| \begin{array}{l} (Z, \mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\}$$

The Gromov-Hausdorff Distance

Definition

For any two compact metric spaces (X, m_X) and (Y, m_Y), we define $\mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y)$ as:

$$\left\{ (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \,\middle|\, \begin{array}{l} (Z, \mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\}$$

Notation

The *Hausdorff distance* on the compact subsets of a metric space (X, m) is denoted by $\mathsf{Haus}_{\mathsf{m}}$.

Definition (Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X, m_X) and (Y, m_Y) is:

$$\inf\left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X), \iota_Y(Y)) : (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \in \mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y) \right\}.$$

A noncommutative Gromov-Hausdorff distance

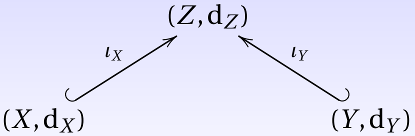

Figure: Gromov-Hausdorff Isometric Embeddings

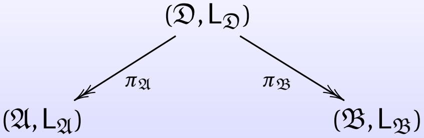

Figure: A tunnel

The Dual Gromov-Hausdorff Propinquity

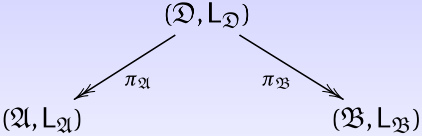

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

The Dual Gromov-Hausdorff Propinquity

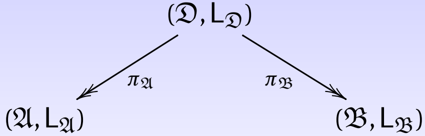

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

Definition (The extend of a tunnel)

The *extent* of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{A}))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}(\mathscr{S}(\mathfrak{B}))\right)\right\}.$$

The Dual Gromov-Hausdorff Propinquity

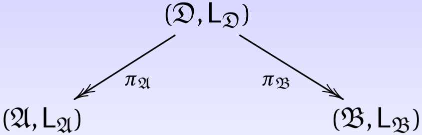

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

Definition (The extend of a tunnel)

The *extent* of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{A}))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}(\mathscr{S}(\mathfrak{B}))\right)\right\}.$$

Definition (L. ,13, 14 / special case)

The *dual propinquity* $\Lambda_F^{*}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\inf\left\{\chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.$$

The Dual Gromov-Hausdorff Propinquity

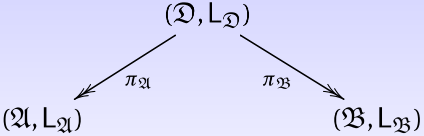

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

Definition (L., 13, 14 / special case)

The *dual propinquity* $\Lambda^*_F((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\inf\left\{\chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.$$

Theorem (L., 13)

The dual propinquity is a *complete metric* up to *full quantum isometry*, which induces the same topology on classical compact metric spaces as the Gromov-Hausdorff distance.

Bridges

Theorem (L. (13))

For any *bridge* $\gamma = (\mathfrak{E}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ where $\mathfrak{E}$ is a unital C*-algebra, $x \in \mathfrak{E}$ and $\pi_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{E}$, $\pi_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{E}$ are unital *-monomorphisms, there exists $\lambda(\gamma) > 0$ such that:

$$(\mathfrak{A} \oplus \mathfrak{B}, (a,b) \mapsto a, (a,b) \mapsto b,$$
$$(a,b) \mapsto \max\left\{ \mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), \frac{1}{\lambda(\gamma)} \|\pi_{\mathfrak{A}}(a)x - x\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}} \right\}$$

is a tunnel of extend at most $2\lambda(\gamma)$.

Bridges

Theorem (L. (13))

For any *bridge* $\gamma = (\mathfrak{E}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ where $\mathfrak{E}$ is a unital C*-algebra, $x \in \mathfrak{E}$ and $\pi_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{E}$, $\pi_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{E}$ are unital *-monomorphisms, there exists $\lambda(\gamma) > 0$ such that:

$$\begin{aligned}&(\mathfrak{A} \oplus \mathfrak{B}, (a,b) \mapsto a, (a,b) \mapsto b,\\ &\qquad (a,b) \mapsto \max\left\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), \frac{1}{\lambda(\gamma)}\|\pi_{\mathfrak{A}}(a)x - x\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}}\right\}\end{aligned}$$

is a tunnel of extend at most $2\lambda(\gamma)$.

The number $\lambda(\gamma)$ is obtained as the maximum of:

$$\begin{aligned}\max\Big\{&\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}}(\mathscr{S}(\mathfrak{A}), \pi_{\mathfrak{A}}^*(\mathscr{S}(\mathfrak{D}|x))),\\ &\qquad\qquad \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}}(\mathscr{S}(\mathfrak{B}), \pi_{\mathfrak{B}}^*(\mathscr{S}(\mathfrak{D}|x)))\Big\},\end{aligned}$$

and $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{D}}}(\{\pi_{\mathfrak{A}}(a)x \in sa(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) \leq 1\}, \{x\pi_{\mathfrak{B}}(b) \in sa(\mathfrak{A}) : \mathsf{L}_{\mathfrak{B}}(b) \leq 1\})$.

Bridges

Theorem (L. (13))

For any *bridge* $\gamma = (\mathfrak{E}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ where $\mathfrak{E}$ is a unital C*-algebra, $x \in \mathfrak{E}$ and $\pi_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{E}$, $\pi_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{E}$ are unital *-monomorphisms, there exists $\lambda(\gamma) > 0$ such that:

$$(\mathfrak{A} \oplus \mathfrak{B}, (a,b) \mapsto a, (a,b) \mapsto b,$$
$$(a,b) \mapsto \max\left\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), \frac{1}{\lambda(\gamma)} \|\pi_{\mathfrak{A}}(a)x - x\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}}\right\}$$

is a tunnel of extend at most $2\lambda(\gamma)$.

We may use the *length* $\inf\{\lambda(\cdot)\}$ of a bridge to construct a distance on quasi-Leibniz quantum compact metric spaces, the *quantum propinquity* Λ, which dominates the dual propinquity, and induces the same topology on classical compact metric spaces.
All proofs of convergence to date for the dual propinquity are in fact done for the stronger quantum propinquity.

Finite Dimensional Approximations of quantum tori

Theorem (L. (13))

If for all $n \in \mathbb{N}$, we set $\mathscr{F}_n = C^*(U_n, V_n) = C^*(\mathbb{Z}_n^2, \rho_n)$ where:

$$U_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & \dots & 1 & 0 \end{pmatrix}, V_n = \begin{pmatrix} 1 & & & & \\ & \rho_n & & & \\ & & \rho_n^2 & & \\ & & & \ddots & \\ & & & & \rho_n^{n-1} \end{pmatrix}$$

with $\rho_n = e^{2i\pi\frac{p_n}{n}} \neq 1$, and if $\lim_{n\to\infty} \rho_n = \rho$, then:

$$\lim_{n\to\infty} \Lambda((\mathscr{F}_n, \mathsf{L}_n), (\mathscr{A}_\rho, \mathsf{L})) = 0$$

where $\mathscr{A}_\rho = C^*(U, V)$ and U, V are universal unitaries such that $VU = \rho UV$, while L_n and L are L-seminorms from the dual actions, and for some *fixed* continuous length function on $\mathbb{T}^2$.

Quantum Tori and the quantum propinquity

Theorem (Latrémolière, 2013)

Let $d \in \mathbb{N} \setminus \{0,1\}$, σ a multiplier of $\mathbb{Z}^d$. For each $n \in \mathbb{N}$, let $k_n \in \overline{\mathbb{N}}^d$ and σ_n be a multiplier of $\mathbb{Z}^d_k = \mathbb{Z}^d / k_n \mathbb{Z}^d$ such that:

1. $\lim_{n\to\infty} k_n = (\infty, \ldots, \infty)$,
2. the unique lifts of σ_n to $\mathbb{Z}^d$ as multipliers converge pointwise to σ.

Then $\lim_{n\to\infty} \Lambda\left(C^*\left(\mathbb{Z}^d, \sigma\right), C^*\left(\mathbb{Z}^d_{k_n}, \sigma_n\right)\right) = 0$, where the Lip-norms are given by the dual actions for any *fixed* length function on $\mathbb{T}^d$.

Quantum Tori and the quantum propinquity

Theorem (Latrémolière, 2013)

Let $d \in \mathbb{N} \setminus \{0,1\}$, σ a multiplier of $\mathbb{Z}^d$. For each $n \in \mathbb{N}$, let $k_n \in \overline{\mathbb{N}}^d$ and σ_n be a multiplier of $\mathbb{Z}_k^d = \mathbb{Z}^d / k_n \mathbb{Z}^d$ such that:

1. $\lim_{n\to\infty} k_n = (\infty, \ldots, \infty)$,
2. the unique lifts of σ_n to $\mathbb{Z}^d$ as multipliers converge pointwise to σ.

Then $\lim_{n\to\infty} \Lambda\left(C^*\left(\mathbb{Z}^d, \sigma\right), C^*\left(\mathbb{Z}^d_{k_n}, \sigma_n\right)\right) = 0$, where the Lip-norms are given by the dual actions for any *fixed* length function on $\mathbb{T}^d$.

Theorem (Aguilar and L. (15) / Informal)

The Effros-Shen AF algebras parametrized by the space of irrational real numbers (i.e. by the Baire space) are a continuous family of quasi-Leibniz quantum compact metric spaces for the quantum propinquity.

Other examples

Example: Other examples of convergence

1. Conformal perturbations of quantum metrics (L., 15)
2. Dabrowsky and Sitarz' Curved quantum tori (L., 15)
3. AF algebras as limits of their inductive sequence in a *metric* sense; UHF algebras and Effros-Shen algebras form continuous families (Aguilar and L., 15),
4. Spheres as limits of full matrix algebras (Rieffel, 15)
5. Nuclear quasi-diagonal quasi-Leibniz quantum compact metric spaces have finite dim approximations (L., 15),
6. There exists an analogue of Gromov's compactness theorem (L., 15)
7. Noncommutative solenoids form a continuous family and have approximations by quantum tori (L. and Packer, 16)
8. Closed balls for the noncommutative Lipschitz distance are totally bounded for Λ (L., 16)

GPS

Metrics for Vector Bundles

What is a mean to encode metric information on vector bundles?

Metrics for Vector Bundles

What is a mean to encode metric information on vector bundles?
Let V be a $\mathbb{C}$-vector bundles over a compact Riemannian manifold M. A *metric* on V is given as a (smooth) section of the associated bundle of sesquilinear products on the fibers of V.
This gives an inner product on the module ΓV of continuous sections of V over M, valued in $C(M)$.

Metrics for Vector Bundles

What is a mean to encode metric information on vector bundles?
Let V be a $\mathbb{C}$-vector bundles over a compact Riemannian manifold M. A *metric* on V is given as a (smooth) section of the associated bundle of sesquilinear products on the fibers of V.
This gives an inner product on the module ΓV of continuous sections of V over M, valued in $C(M)$.
Moreover, we always have a *metric connection* ∇, such that:

$$d_X g(\omega,\eta) = g(\nabla_X \omega,\eta) + g(\omega,\nabla_X \eta).$$

Metrics for Vector Bundles

What is a mean to encode metric information on vector bundles?
Let V be a $\mathbb{C}$-vector bundles over a compact Riemannian manifold M. A *metric* on V is given as a (smooth) section of the associated bundle of sesquilinear products on the fibers of V.
This gives an inner product on the module ΓV of continuous sections of V over M, valued in $C(M)$.
Moreover, we always have a *metric connection* ∇, such that:

$$d_X g(\omega,\eta) = g(\nabla_X \omega,\eta) + g(\omega,\nabla_X \eta).$$

We propose that *both* g and $|||\nabla\cdot|||$ contain metric information.

Metrized quantum vector bundles

Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is given by:

1. $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space,
2. $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ is a left Hilbert module over $\mathfrak{A}$,
3. D is a norm on a dense subspace of $\mathscr{M}$ such that:
    1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
    2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
    3. $\mathsf{D}(a\omega) \leqslant G(\|a\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{D}(\omega))$,
    4. $\mathsf{L}(\langle\omega,\eta\rangle_{\mathscr{M}}) \leqslant H(\mathsf{D}(\omega), \mathsf{D}(\eta))$.

Metrized quantum vector bundles

Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is given by:

1. $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space,
2. $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ is a left Hilbert module over $\mathfrak{A}$,
3. D is a norm on a dense subspace of $\mathscr{M}$ such that:
   1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
   2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
   3. $\mathsf{D}(a\omega) \leqslant G(\|a\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{D}(\omega))$,
   4. $\mathsf{L}(\langle\omega,\eta\rangle_{\mathscr{M}}) \leqslant H(\mathsf{D}(\omega), \mathsf{D}(\eta))$.

Example: Classical picture

For a compact Riemannian manifold M and a $\mathbb{C}$-vector bundle V with a metric g and metric connection ∇, we use the inner product $\langle\omega,\eta\rangle_g = \int_M g_x(\omega_x, \eta_x)\, d\mathrm{Vol}(x)$ and $\mathsf{D}(\omega) = \max\{\|\omega\|_g, |||\nabla.\omega|||\}$.

Metrized quantum vector bundles

Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is given by:

1. $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space,
2. $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ is a left Hilbert module over $\mathfrak{A}$,
3. D is a norm on a dense subspace of $\mathscr{M}$ such that:
   1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
   2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
   3. $\mathsf{D}(a\omega) \leqslant G(\|a\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{D}(\omega))$,
   4. $\mathsf{L}(\langle\omega,\eta\rangle_{\mathscr{M}}) \leqslant H(\mathsf{D}(\omega), \mathsf{D}(\eta))$.

Example: Free modules

Given $(\mathfrak{A}, \mathsf{L})$, we set $\langle(a_1,\ldots,a_d),(b_1,\ldots,b_d)\rangle_d = \sum_{j=1}^d a_j b_j^*$ and $\mathsf{L}_d(a_1,\ldots,a_d) = \max\left\{\mathsf{L}(\Re a_j), \mathsf{L}(\Im a_j) : j \in \{1,\ldots,d\}\right\}$. Let $\mathsf{D} = \max\{\|\cdot\|_d, \mathsf{L}_d\}$. Then $(\mathfrak{A}^d, \langle,\rangle_d, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is a metrized quantum vector bundle.

Metrized quantum vector bundles

Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is given by:

1. $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space,
2. $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ is a left Hilbert module over $\mathfrak{A}$,
3. D is a norm on a dense subspace of $\mathscr{M}$ such that:
   1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
   2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
   3. $\mathsf{D}(a\omega) \leqslant G(\|a\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{D}(\omega))$,
   4. $\mathsf{L}(\langle\omega,\eta\rangle_{\mathscr{M}}) \leqslant H(\mathsf{D}(\omega), \mathsf{D}(\eta))$.

Full quantum isometries

(θ, Θ) full quantum isometry when θ full quantum isometry between bases and $\Theta(a\xi) = \theta(a)\Theta(\xi)$, Θ linear isomorphism preserving both the norms and the D-norms.

The Heisenberg Modules (Connes, 81; Rieffel)

Fix $\theta \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $d \in \mathbb{N} \setminus \{0\}$ such that $\eth = \theta - \frac{p}{q} \neq 0$.

The Heisenberg Modules (Connes, 81; Rieffel)

Fix $\theta \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $d \in \mathbb{N} \setminus \{0\}$ such that $\eth = \theta - \frac{p}{q} \neq 0$.

1. Start with a representation of $\left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R}^3 \right\}$ on $L^2(\mathbb{R})$:

$$\alpha_{\eth}^{x,y,t} \xi(s) = \exp(i\pi(t + 2xs))\xi(s + \eth y).$$

Promote it to $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.

The Heisenberg Modules (Connes, 81; Rieffel)

Fix $\theta \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $d \in \mathbb{N} \setminus \{0\}$ such that $\eth = \theta - \frac{p}{q} \neq 0$.

1. Start with a representation of $\left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R}^3 \right\}$ on $L^2(\mathbb{R})$:
$$\alpha_{\eth}^{x,y,t}\xi(s) = \exp(i\pi(t + 2xs))\xi(s + \eth y).$$
Promote it to $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.
2. Let $W_1, W_2 \in U(d)$ with $W_1 W_2 = e^{2i\pi p/q} W_2 W_1$ and $W_1^n = W_2^n = 1$. We get a $\mathscr{A}_\theta = C^*(u_\theta, v_\theta)$-module with:
$$(u_\theta^n v_\theta^m)\xi = W_1^n W_2^m \alpha_{\eth}^{n,m,0}\xi.$$

The Heisenberg Modules (Connes, 81; Rieffel)

Fix $\theta \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $d \in \mathbb{N} \setminus \{0\}$ such that $\eth = \theta - \frac{p}{q} \neq 0$.

1. Start with a representation of $\left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R}^3 \right\}$ on $L^2(\mathbb{R})$:
$$\alpha_{\eth}^{x,y,t}\xi(s) = \exp(i\pi(t + 2xs))\xi(s + \eth y).$$
Promote it to $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.
2. Let $W_1, W_2 \in U(d)$ with $W_1 W_2 = e^{2i\pi p/q} W_2 W_1$ and $W_1^n = W_2^n = 1$. We get a $\mathscr{A}_\theta = C^*(u_\theta, v_\theta)$-module with:
$$(u_\theta^n v_\theta^m)\xi = W_1^n W_2^m \alpha_{\eth}^{n,m,0}\xi.$$
3. For Schwarz functions ξ, ω, set:
$$\langle \xi, \omega \rangle_{\mathscr{H}_\theta^{p,q,d}} = \sum_{n,m \in \mathbb{Z}} \langle u_\theta^n v_\theta^m \xi, \omega \rangle_{L^2(\mathbb{R},\mathbb{C}^d)} u_\theta^n v_\theta^m;$$
complete space of Schwarz functions to the *Heisenberg module* $\mathscr{H}_\theta^{p,q,d}$.

The D-norm

Definition (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\| \alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi \right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

The D-norm

Definition (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\| \alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi \right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

Theorem (L., 16)

$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

The D-norm

Definition (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\| \alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi \right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

Theorem (L., 16)

$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

The D-norm, when restricted to Schwarz functions, is the *norm of a connection studied by Connes, Rieffel for the Yang Mills problem on the quantum 2-torus.* The connection arises from the infinitesimal rep of the Heisenberg Lie algebra form $\alpha_\eth$.

The D-norm

Definition (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

Theorem (L., 16)

$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

The proof rests on the compactness, for f radial, of:

$$\xi \longmapsto \alpha_{\eth}^{f}\xi = \iint_{\mathbb{R}^2} f(x,y)\alpha^{x,y,\frac{xy}{2}}\xi \, dx dy$$

The D-norm

Definition (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

Theorem (L., 16)

$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

For f a linear combination of certain *Laguerre functions*, $\alpha_{\eth}^{f}$ is finite rank.

The D-norm

Definition (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

Theorem (L., 16)

$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

We can fabricate an L^1-approximate unit of linear combinations of Laguerre functions using *Thangavelu* results on *Césaro sums of Laguerre expansions*.

The D-norm

Definition (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\| \alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi \right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

Theorem (L., 16)

$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

We then check
$\|\xi - \alpha_{\eth}^{f}(\xi)\|_{\mathscr{H}_\theta^{p,q,d}} \leqslant \mathsf{D}_\theta^{p,q,d}(\xi) \iint f(x,y)(2\pi|\eth|\|x,y\|)\, dx dy.$

The D-norm

Definition (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

Theorem (L., 16)

$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

Many details involved which are different from Rieffel's ergodic action result (which relied on finite dimension of spectral subspaces).

GPS

Bridges for modules

Fix $\Omega_{\mathfrak{A}} = (\mathscr{M}_{\mathfrak{A}}, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $\Omega_{\mathfrak{B}} = (\mathscr{M}_{\mathfrak{B}}, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two metrized quantum vector bundles.

Definition (L., 16)

A *modular bridge* $\left(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j)_{j\in J}, (\eta_j)_{j\in J}\right)$ is a bridge $(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ and two families $(\omega_j)_{j\in J} \in \mathscr{M}_{\mathfrak{A}}$, $(\eta_j)_{j\in J} \in \mathscr{M}_{\mathfrak{B}}$ with $\mathsf{D}_{\mathfrak{A}}(\omega_j), \mathsf{D}_{\mathfrak{B}}(\eta_j) \leqslant 1$ for all $j \in J$.

Bridges for modules

Fix $\Omega_{\mathfrak{A}} = (\mathscr{M}_{\mathfrak{A}}, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $\Omega_{\mathfrak{B}} = (\mathscr{M}_{\mathfrak{B}}, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two metrized quantum vector bundles.

Definition (L., 16)

A *modular bridge* $\left(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j)_{j\in J}, (\eta_j)_{j\in J}\right)$ is a bridge $(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ and two families $(\omega_j)_{j\in J} \in \mathscr{M}_{\mathfrak{A}}$, $(\eta_j)_{j\in J} \in \mathscr{M}_{\mathfrak{B}}$ with $\mathsf{D}_{\mathfrak{A}}(\omega_j), \mathsf{D}_{\mathfrak{B}}(\eta_j) \leqslant 1$ for all $j \in J$.

Definition (L., 16)

The *length* of a modular bridge is the maximum of the length of its basic bridge, and the sum of:

1. the maximum of $\mathsf{Haus}_k(\{\omega_j : j \in J\}, \{\omega : \mathsf{D}_{\mathfrak{A}}(\omega) \leqslant 1)$ and its counterpart in $\Omega_{\mathfrak{B}}$, where:

$$k(\omega,\xi) = \sup\left\{\|\langle\omega,\eta\rangle_{\mathfrak{A}} - \langle\xi,\eta\rangle_{\mathfrak{A}}\|_{\mathfrak{A}} : \mathsf{D}_{\mathfrak{A}}(\eta) \leqslant 1\right\},$$

2. $\max\left\{\|\pi_{\mathfrak{A}}(\langle\omega_j,\omega_k\rangle_{\mathfrak{A}})x - x\pi_{\mathfrak{B}}(\langle\eta_j,\eta_k\rangle_{\mathfrak{B}})\|_{\mathfrak{D}} : j \in J\right\}$.

The modular propinquity

Definition (L., 16)

The *modular propinquity* is the largest pseudo-metric Λ^{mod} such that $\Lambda^{\mathsf{mod}}(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}) \leqslant \lambda(\gamma)$ for any modular γ from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$.

The modular propinquity

Definition (L., 16)

The *modular propinquity* is the largest pseudo-metric Λ^{mod} such that $\Lambda^{\mathsf{mod}}(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}) \leqslant \lambda\left(\gamma\right)$ for any modular γ from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$.

Theorem (L., 16)

The *modular propinquity* is a metric on metrized quantum vector bundles up to full quantum isometry.

The modular propinquity

Definition (L., 16)

The *modular propinquity* is the largest pseudo-metric Λ^{mod} such that $\Lambda^{\mathsf{mod}}(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}) \leqslant \lambda\left(\gamma\right)$ for any modular γ from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$.

Theorem (L., 16)

The *modular propinquity* is a metric on metrized quantum vector bundles up to full quantum isometry.

Theorem (Free modules; L., 16)

If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are quasi-Leibniz quantum compact metric space then:

$$\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant$$
$$\Lambda^{\mathsf{mod}}((\mathfrak{A}^n, \mathsf{D}^n_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}^n, \mathsf{D}^n_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant 2n\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$$

where $\mathsf{D}^n_{\mathfrak{A}}(a_1, \ldots, a_n) = \max_{j=1,\ldots,n}\{\|a_j\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}(\Re(a_j)), \mathsf{L}_{\mathfrak{A}}(\Im(a_j))\}$.

Theorem (L., 16)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and p, q, d fixed. If for all $\theta \in \mathbb{R}$, and $a \in \mathscr{A}_\theta$:

$$\mathsf{L}_\theta(a) = \sup\left\{ \frac{\left\|\beta_\theta^{\exp(ix),\exp(iy)} a - a\right\|_{\mathscr{A}_\theta}}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where β_θ is the dual action, and for all $\xi \in \mathscr{H}_\theta^{p,q,d}$ we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}} \xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where $\eth = \theta - p/q$, then:

$$\lim_{\vartheta \to \theta} \Lambda^{\mathsf{mod}}\left(\left(\mathscr{H}_\vartheta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\vartheta^{p,q,d}}, \mathsf{D}_\vartheta^{p,q,d}, \mathscr{A}_\vartheta, \mathsf{L}_\vartheta\right), \left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)\right) = 0.$$