Convergence of Quantum Metric Spaces

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Noncommutative Metric Geometry

Founding Allegory of Noncommutative Metric Geometry

Noncommutative *metric* geometry is the study of noncommutative generalizations of algebras of *Lipschitz* functions over *metric* spaces.

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Noncommutative *metric* geometry is the study of noncommutative generalizations of algebras of *Lipschitz* functions over *metric* spaces.

- Pioneered by *Rieffel (1998–)*, inspired by *Connes (1989)*.
- Motivated by mathematical physics, addresses problems such as:
  - Can we approximate quantum spaces with finite dimensional algebras?
  - Are certain functions from a topological space to quantum spaces continuous? Lipschitz?
  - Are certain functions from a topological space to modules over quantum spaces continuous?

Structure of the talk

1. Compact Quantum Metric Spaces
2. Convergence of quasi-Leibniz quantum compact metric space
3. Gromov's Compactness Theorem
4. The Modular Propinquity

GPS

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\mathsf{L}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{d}(x,y)} : x, y \in X, x \neq y \right\}$$

for all $f \in C(X)$ (allowing ∞).

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The *Monge-Kantorovich metric* on $\mathscr{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

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The Gelfand map $x \in (X, \mathsf{m}) \mapsto \delta_x \in (\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}})$ is an isometry.

Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

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We call L an *L-seminorm*.

Lipschitz morphisms and Quantum Isometries

Theorem-Definition (Lipschitz Morphisms)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *k-Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that any of the following equivalent statement holds:

1. $\varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$ is a *k-Lipschitz map from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ to $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$*,
2. (Rieffel, 00) $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant k \mathsf{L}_{\mathfrak{A}}$,
3. (L., 16) $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

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Definition (Rieffel (98), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that:

$$\forall b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) \quad \mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.$$

A *full quantum isometry* π is a *-isomorphism such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

Ergodic actions of compact metric groups

Theorem (Rieffel, 98)

Let G be a *compact group* endowed with a *continuous length function* ℓ. Let α be an *action* of G on some *unital C*-algebra* $\mathfrak{A}$. Set:

$$\forall a \in \mathfrak{A} \quad \mathsf{L}(a) = \sup \left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1_G\} \right\}.$$

$(\mathfrak{A}, \mathsf{L})$ is a *Leibniz quantum compact metric space* if and only if $\left\{ a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a \right\} = \mathbb{C}1_{\mathfrak{A}}$.

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Example: Quantum tori

- $G = \mathbb{T}^d$,
- $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$ (universal for $U_j U_k = \sigma(j,k) U_{j+k}$),
- α : dual action ($\alpha^z U_j = z^j U_j$).
- Associated with a differential calculus when ℓ from invariant Riemannian metric.

Spectral triples and quantum metrics

Theorem (L., 15)

Let α be an *ergodic action* of a *compact Lie group* G on a unital C*-algebra $\mathfrak{A}$. Let π be the GNS rep from the invariant tracial state of $\mathfrak{A}$. Let $(X_1, \ldots, X_n)$ be an orthonormal basis for the *Lie algebra* $\mathfrak{g}$ of G equipped with an inner product, and let:

$$\partial_j : a \in \mathfrak{A}^1 \mapsto \lim_{t \to 0} \frac{\alpha^{\exp(tX_j)} a - a}{t} \text{ for } j \in \{1, \ldots, n\}.$$

For $H = \begin{pmatrix} h_{11} & \ldots & h_{1n} \\ \vdots & & \vdots \\ h_{n1} & \ldots & h_{nn} \end{pmatrix} \in M_n(\pi(\mathfrak{A})')$, we set:

$$D_H = \sum_{j=1}^{n} \sum_{k=1}^{n} h_{jk} \partial_k \otimes \mathrm{Clifford}(X_j)$$

$\left(\mathfrak{A}, \left|\left|\left| [D_H, \pi^{\oplus n}(\cdot)] \right|\right|\right|\right)$ is a *Leibniz quantum compact metric space*.

AF algebras with tracial state

Theorem (Aguilar, L., 15)

- Let $\mathfrak{A} = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$ be an *AF-algebra with a faithful tracial state* τ and where $\dim \mathfrak{A}_n < \infty$ for all $n \in \mathbb{N}$.
- For all $n \in \mathbb{N}$ let $\mathbb{E}_n : \mathfrak{A} \to \mathfrak{A}_n$ be the unique conditional expectation with $\tau \circ \mathbb{E}_n = \tau$.
- Let $(\beta_n)_{n\in\mathbb{N}}$ in $(0,\infty)^{\mathbb{N}}$, with limit 0.

If, for all $a \in \mathfrak{sa}(\mathfrak{A})$, we set:

$$\mathsf{L}(a) = \sup\left\{ \frac{\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}}}{\beta_n} : n \in \mathbb{N} \right\}$$

then $(\mathfrak{A}, \mathsf{L})$ is a $(2,0)$–quasi-Leibniz quantum compact metric space.

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The quasi-Leibniz condition here is:

$$\forall a, b \in \mathfrak{A} \quad \max\{\mathsf{L}(a \circ b), \mathsf{L}(\{a,b\})\} \leqslant 2(\mathsf{L}_{\mathfrak{A}}(a)\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}}\mathsf{L}_{\mathfrak{A}}(b)).$$

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This construction recovers the usual ultrametrics of the Cantor set.

Some Other Examples

1. Hyperbolic group C*-algebras (Rieffel, Ozawa, 05),
2. Nilpotent group C*-algebras (Christ, Rieffel, 16),
3. Connes-Landi spheres (Li, 03)
4. Conformal deformations of quantum metric spaces from spectral triples (L., 15)
5. Group C*-algebras for groups with rapid decay (Antonescu, Christensen, 2004)
6. Noncommutative Solenoids (L., Packer, 16)
7. Certain C*-crossed-products (J. Bellissard, M. Marcolli, Reihani, 10), (involves my work on locally compact quantum metric space).

GPS

The Gromov-Hausdorff Distance

Definition

For any two compact metric spaces (X, m_X) and (Y, m_Y), we define $\mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y)$ as:

$$\left\{ (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \middle| \begin{array}{l} (Z, \mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\}$$

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Notation

The *Hausdorff distance* on the compact subsets of a metric space (X, m) is denoted by $\mathsf{Haus}_{\mathsf{m}}$.

Definition (Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X, m_X) and (Y, m_Y) is:

$$\inf\left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X), \iota_Y(Y)) : (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \in \mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y) \right\}.$$

A noncommutative Gromov-Hausdorff distance

Problem

How to generalize Gromov's construction to quasi-Leibniz quantum compact metric spaces ?

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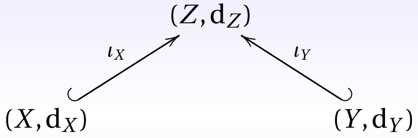

Figure: Isometric Embeddings

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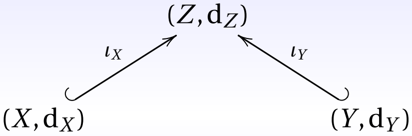

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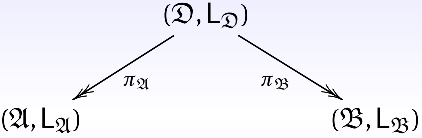

Figure: A tunnel

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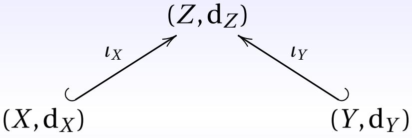

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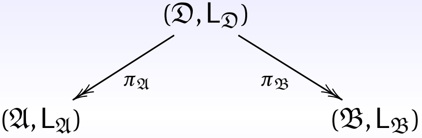

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Choices

- What class $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$ should belong to? *Should we assume a form of quasi-Leibniz inequality?*
- What kind of morphisms $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ should we choose?
- How do we quantify a bridge?

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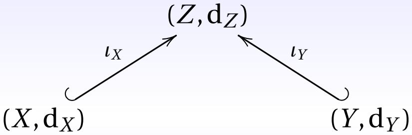

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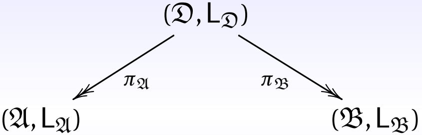

Figure: A tunnel

Previous problems

- *Coincidence problem:* distance zero may not imply *-isomorphism (Rieffel, Wu)
- *Triangle Inequality:* working with quasi-Leibniz $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$ meant abandoning the triangle inequality (Kerr, Rieffel)

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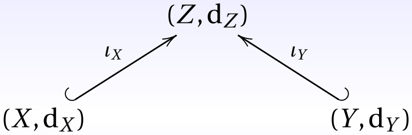

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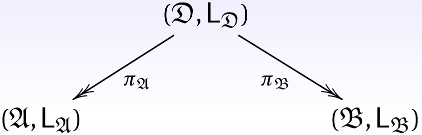

Figure: A tunnel

Previous problem root cause

Noncommutative Gromov-Hausdorff distances construction went out of the category of quasi-Leibniz quantum compact metric spaces. Thus, no easy applications to modules, morphisms, ...

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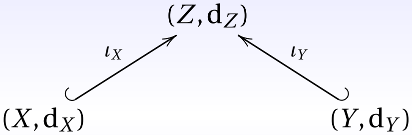

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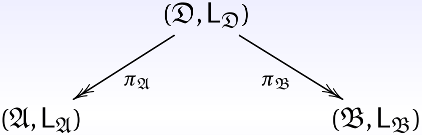


Figure: A tunnel

Solution (L., 13)

- We choose isometric embeddings which are **-morphisms*.
- We restrict embeddings to *quasi-Leibniz quantum compact metric spaces* or even more specific if desired.

The Dual Gromov-Hausdorff Propinquity

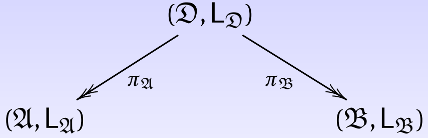
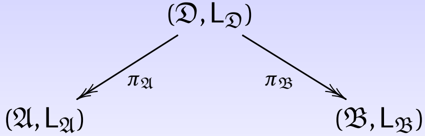


Figure: An F-tunnel: all spaces are F-quasi-Leibniz

The Dual Gromov-Hausdorff Propinquity



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Definition (The extent of a tunnel)

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{A}))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}(\mathscr{S}(\mathfrak{B}))\right)\right\}.$$

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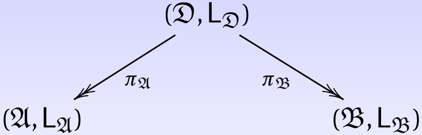

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Definition (L. ,13, 14 / special case)

The *dual propinquity* $\Lambda_F^{*}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\inf\left\{\chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.$$

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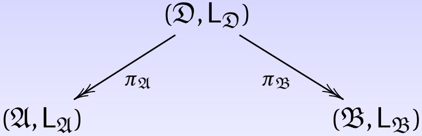

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Definition (L., 13, 14 / special case)

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Theorem (L., 13)

The dual propinquity is a *complete metric* up to *full quantum isometry*, which induces the same topology on classical compact metric spaces as the Gromov-Hausdorff distance.

Quantum Tori and the quantum propinquity

Endow $\mathbb{T}^d$ with a continuous length function ℓ. Let α be the dual action of $\widehat{\mathbb{Z}_k} \subseteq \mathbb{T}^d$ on $C^*(\mathbb{Z}^d_k, \sigma)$, and set for $a \in \mathfrak{sa}\left(\mathbb{C}^*(\mathbb{Z}^d_k, \sigma)\right)$:

$$\mathsf{L}_\sigma(a) = \sup\left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in \widehat{\mathbb{Z}^d_k} \setminus \{1\} \right\}.$$

L_σ is an L-seminorm (Rieffel, 98).

Theorem (Latrémolière, 2013)

Let $d \in \mathbb{N} \setminus \{0,1\}$, σ a multiplier of $\mathbb{Z}^d$. If for each $n \in \mathbb{N}$, we let $k_n \in \overline{\mathbb{N}}^d$ and σ_n be a multiplier of $\mathbb{Z}^d_k = \mathbb{Z}^d / k_n \mathbb{Z}^d$ such that:

1. $\lim_{n\to\infty} k_n = (\infty, \ldots, \infty)$,
2. the unique lifts of σ_n to $\mathbb{Z}^d$ as multipliers converge pointwise to σ,

then $\lim_{n\to\infty} \Lambda\left(\left(C^*\left(\mathbb{Z}^d, \sigma\right), \mathsf{L}_\sigma\right), \left(C^*\left(\mathbb{Z}^d_{k_n}, \sigma_n\right), \mathsf{L}_{\sigma_n}\right)\right) = 0.$

Curved Quantum Tori

Theorem (L., 15)

- Let σ be a multiplier of $\mathbb{Z}^d$, and $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$.
- Identify $\mathfrak{A}$ with its image by the regular representation acting on $L^2(\mathfrak{A}, \tau)$.
- Define the length function $\ell : K \in \mathrm{GL}_n(\mathfrak{A}') \mapsto |||1 - K|||_{L^2(\mathfrak{A},\tau)}$.

If $H \in \mathrm{GL}_n(\mathfrak{A}')$ then:

$$\lim_{\substack{G \to H \\ G \in \mathfrak{G}\mathfrak{L}_n(\mathfrak{A}')}} \Lambda((\mathfrak{A}, \mathsf{L}_G), (\mathfrak{A}, \mathsf{L}_H)) = 0$$

where $\mathsf{L}_H = |||[D_H, \cdot]|||$ with $D_H = \sum_{j=1}^n \sum_{k=1}^n H_{jk} \partial_k \otimes X_j$ for a fixed orthonormal basis $(X_1, \ldots, X_d)$ of $\mathbb{R}^d$ seen as the Lie algebra of $\mathbb{T}^d$.

Effros-Shen AF algebras

Theorem (Aguilar, L., 15)

- For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $\theta = \lim_{n\to\infty} \frac{p_n^\theta}{q_n^\theta}$ with $\frac{p_n^\theta}{q_n^\theta} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$

  for $a_1, \ldots \in \mathbb{N}$.
- Set $\mathfrak{AF}_\theta = \varinjlim_{n\to\infty} \left(\mathfrak{M}_{q_n} \oplus \mathfrak{M}_{q_{n-1}}, \psi_{n,\theta}\right)$ where $\psi_{n,\theta}$ involves a_{n+1}.
- For all $n \in \mathbb{N}$, let $\beta_n = \frac{1}{q_n^2 + q_{n-1}^2}$ and L_θ the L-seminorm for this data.

For all $\theta \in \mathbb{R} \setminus \mathbb{Q}$, we have:

$$\lim_{\substack{\vartheta\to\theta \\ \vartheta\notin\mathbb{Q}}} \Lambda\left((\mathfrak{AF}_\vartheta, \mathsf{L}_\vartheta), (\mathfrak{AF}_\theta, \mathsf{L}_\theta)\right) = 0.$$

Other examples

1. Conformal perturbations of quantum metrics (L., 15)
2. AF algebras as limits of their inductive sequence in a *metric* sense; UHF and Effros-Shen algebras form continuous families (Aguilar and L., 15),
3. Spheres as limits of full matrix algebras (Rieffel, 15)
4. Nuclear quasi-diagonal quasi-Leibniz quantum compact metric spaces have finite dim approximations (L., 15),
5. There exists an analogue of Gromov's compactness theorem (L., 15)
6. Noncommutative solenoids form a continuous family and have approximations by quantum tori (L. and Packer, 16)
7. Closed balls for the noncommutative Lipschitz distance are totally bounded for Λ (L., 16)

GPS

The Lipschitz distance

Definition (Lipchitz distance, L. 16)

The *Lipschitz distance* between two quasi-Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is:

$$\mathrm{LipD}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = \inf\Big\{|\ln(\mathrm{dil}(\varphi)|, |\ln(\mathrm{dil}(\varphi^{-1}))| \Big| \\ \varphi : \mathfrak{A} \to \mathfrak{B} \text{ *-isomorphism}\Big\}$$

where $\mathrm{dil}(\varphi)$ is the norm of φ for the L-seminorms $\mathsf{L}_{\mathfrak{A}}$ and $\mathsf{L}_{\mathfrak{B}}$.

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The Lipschitz distance is a *complete extended metric* on the class of quasi-Leibniz quantum compact metric spaces up to full quantum isometry which dominates Λ.

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Problem

The Lipschitz distance is ∞ between non *-isomorphic spaces.

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Compactness of closed Balls (L. 16)

Closed balls for the Lipschitz distance are *compact* for the dual propinquity.

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Applications (L., 16)

Conformal perturbations and *curved quantum tori* form *totally bounded classes* for the quantum propinquity.

Compactness and Effros-Shen Algebras

Let $\mathfrak{AF}_\theta$ be the Effros-Shen algebra constructed from $\theta \in (0,1) \setminus \mathbb{Q}$, endowed with the L-seminorm from its natural inductive limit.

Theorem (Aguilar, L., 2015)

Let C, D be two sequences of natural numbers. The class:

$$\left\{ \mathfrak{AF}_\theta \middle| \begin{array}{l} \theta = \lim_{n\to\infty} \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cfrac{1}{\cdots + \frac{1}{r_n}}}} \\ \forall n \in \mathbb{N} \quad C(n) \leqslant r_n \leqslant D(n) \end{array} \right\}$$

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is *compact* for the quantum propinquity. Its topology is thus the same for the quantum and the dual propinquity.

The proof follows from the characterization of compact subsets of the *Baire space*.

Gromov's compactness Theorem

What is a noncommutative analogue of the following?

Theorem (Gromov, 1981)

A class $\mathscr{S}$ of compact metric spaces is totally bounded for the Gromov-Hausdorff distance if, and only if the following two assertions hold:

1. there exists $D \geqslant 0$ such that for all $(X, \mathsf{m}) \in \mathscr{S}$, the diameter of (X, m) is less or equal to D,
2. there exists a function $G : (0, \infty) \to \mathbb{N}$ such that for every $(X, \mathsf{m}) \in \mathscr{S}$, and for every $\varepsilon > 0$, the smallest number $\mathrm{Cov}_{(X,\mathsf{m})}(\varepsilon)$ of balls of radius ε needed to cover (X, m) is no more than $G(\varepsilon)$.

Since the Gromov-Hausdorff distance is complete, a class of compact metric spaces is compact for the Gromov-Hausdorff distance if and only if it is closed and totally bounded.

A Compactness Theorem for Λ_F

Definition (Latrémolière, 2015)

Let F be a permissible function and $\mathscr{Q}$ the class of F–quasi-Leibniz quantum compact metric spaces. If $(\mathfrak{A}, \mathsf{L}) \in \mathscr{Q}$ and $\varepsilon > 0$ then we define the covering number $\mathrm{cov}_{(F)}(\mathfrak{A}, \mathsf{L}|\varepsilon)$:

$$\min\left\{\dim \mathfrak{B} : (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathscr{Q}, \Lambda_F((\mathfrak{A}, \mathsf{L}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \varepsilon\right\}.$$

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Theorem (Latrémolière, 2015)

Let F be a continuous, permissible function. Let $\mathcal{F}$ be the closure of the class of *finite dimensional* F–quasi-Leibniz quantum compact metric spaces for Λ_F. A subclass $\mathcal{A}$ of $\mathcal{F}$ is *totally bounded* for Λ_F if and only if there exists $G : [0, \infty) \to \mathbb{N}$ and $K \geqslant 0$ such that for all $(\mathfrak{A}, \mathsf{L}) \in \mathcal{A}$:

- $\mathrm{diam}(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}}) \leqslant K$,
- $\mathrm{cov}_{(F)}(A, \mathsf{L}|\varepsilon) \leqslant G(\varepsilon)$.

Totally bounded classes of AF algebras

An application of the noncommutative Gromov's compactness theorem is a description of totally bounded classes of AF algebras based upon the growth rate of their finite dimensional components:

Theorem (Aguilar, L. 15)

Let D, U be two increasing sequences in $\mathbb{N} \setminus \{0\}$ with $D \leqslant U$. The class:

$$\left\{ (\mathfrak{A}, \mathsf{L}) = \varinjlim_{n\to\infty} \mathfrak{A}_n \;\middle|\; \begin{array}{l} \forall n \in \mathbb{N} \quad D_n \leqslant \dim \mathfrak{A}_n \leqslant U_n \\ \mathfrak{A}_0 = \mathbb{C} \\ \exists \mu \in \mathscr{S}(\mathfrak{A}) \quad \mu \text{ tracial, faithful} \\ \mathsf{L}(a) = \sup_{n\in\mathbb{N}} \frac{\|a - \mathbb{E}_n^{\mu}(a)\|_{\mathfrak{A}}}{(\dim \mathfrak{A}_n)^{-1}} \end{array} \right\}$$

is *totally bounded* for the quantum propinquity.

What can we find in the closure of f.d. quasi-Leibniz quantum compact metric spaces?

Definition (Latrémolière, 2015)

A unital C*-algebra $\mathfrak{A}$ is *pseudo-diagonal* when, for all $\varepsilon > 0$ and for all finite subset $\mathfrak{F}$ of $\mathfrak{A}$, there exists a finite dimensional C*-algebra $\mathfrak{B}$ and two unital, positive linear maps $\varphi : \mathfrak{B} \to \mathfrak{A}$ and $\psi : \mathfrak{A} \to \mathfrak{B}$ such that for all $a, b \in \mathfrak{F}$:

- $\|a - \varphi \circ \psi(a)\|_{\mathfrak{A}} \leqslant \varepsilon$,
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Theorem (Latrémolière, 2015)

Every unital, nuclear, quasi-diagonal C*-algebra is pseudo-diagonal.

This uses a result from Blackadar and Kirchberg.

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Notation (Latrémolière, 2015)

If $C \geqslant 1$, $D \geqslant 0$, a quasi-Leibniz quantum compact metric space $(\mathfrak{A}, \mathsf{L})$ such that for all $a, b \in \mathfrak{sa}(\mathfrak{A})$:

$$\mathsf{L}(a \circ b) \vee \mathsf{L}(\{a, b\}) \leqslant C(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{B}}) + D\mathsf{L}(a)\mathsf{L}(b)$$

is a (C, D)–quasi-Leibniz quantum compact metric space.

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Theorem (Latrémolière, 2015)

Let $C \geqslant 1$, $D \geqslant 0$. If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is a (C, D)–quasi-Leibniz quantum compact metric space with $\mathfrak{A}$ pseudo-diagonal, and if $\delta > 0$, then there exists a sequence $(\mathfrak{B}_n, \mathsf{L}_n)_{n\in\mathbb{N}}$ of *finite dimensional $(C+\delta, D+\delta)$–quasi-Leibniz quantum compact metric spaces* such that $\lim_{n\to\infty} \Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}_n, \mathsf{L}_n)) = 0$.

GPS

Metrics for Vector Bundles

- Let (M, g) be a compact, connected Riemannian manifold, and let (V, h) be a vector bundle endowed with a metric.

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$$\langle \omega, \xi \rangle_{C(M)} : x \in M \longmapsto h_x(\omega_x, \xi_x) \in C(M).$$

Thus, ΓV is a $C(M)$-Hilbert module.

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- Let ∇ be a metric connection on ΓV, i.e.:

$$d_X \langle \omega, \xi \rangle = \langle \nabla_X \omega, \xi \rangle + \langle \omega, \nabla_X \xi \rangle.$$

  ∇ defines a norm on a dense subspace of ΓV:

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  Our idea is to introduce a metric on objects of the form $(\Gamma V, \langle \cdot, \cdot \rangle_{C(M)}, \mathsf{D}, C(M), \mathsf{L})$.

Metrized quantum vector bundles

Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is given by:

1. $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space,
2. $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ is a left Hilbert module over $\mathfrak{A}$,
3. D is a norm on a dense subspace of $\mathscr{M}$ such that:
   1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
   2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
   3. $\mathsf{D}(a\omega) \leqslant G(\|a\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{D}(\omega))$,
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Full quantum isometries

(θ, Θ) full quantum isometry when θ full quantum isometry between bases and $\Theta(a\xi) = \theta(a)\Theta(\xi)$, Θ linear isomorphism preserving both the norms and the D-norms.

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    4. $\mathsf{L}(\langle\omega,\eta\rangle_{\mathscr{M}}) \leqslant H(\mathsf{D}(\omega), \mathsf{D}(\eta))$.

Example: Classical picture

Hermitian bundles over compact connected Riemannian manifolds.

Metrized quantum vector bundles

Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is given by:

1. $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space,
2. $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ is a left Hilbert module over $\mathfrak{A}$,
3. D is a norm on a dense subspace of $\mathscr{M}$ such that:
   1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
   2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
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Example: Free modules

Given $(\mathfrak{A}, \mathsf{L})$, we set $\langle(a_1,\ldots,a_d),(b_1,\ldots,b_d)\rangle_d = \sum_{j=1}^d a_j b_j^*$ and $\mathsf{L}_d(a_1,\ldots,a_d) = \max\left\{\mathsf{L}(\Re a_j), \mathsf{L}(\Im a_j) : j \in \{1,\ldots,d\}\right\}$. Let $\mathsf{D} = \max\{\|\cdot\|_d, \mathsf{L}_d\}$. Then $(\mathfrak{A}^d, \langle,\rangle_d, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is a metrized quantum vector bundle.

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Example: Heisenberg Modules

Heisenberg modules and their natural *connection*, as build by Connes (81), are (non-free, finitely generated, projective) metrized quantum vector bundles.

The modular Propinquity

Theorem-Definition (The Modular Propinquity (L.,16))

There exists a *distance Λ^{mod}, up to full isometry,* on the class of metrized quantum vector bundles, whose restriction to quasi-Leibniz quantum compact metric spaces, is equivalent to the quantum propinquity.

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Theorem (Free Modules (L., 17))

If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are quasi-Leibniz quantum compact metric space then:

$$\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant$$
$$\Lambda^{\mathsf{mod}}((\mathfrak{A}^n, \mathsf{D}^n_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}^n, \mathsf{D}^n_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant 2n\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$$

where $\mathsf{D}^n_{\mathfrak{A}}(a_1, \ldots, a_n) = \max_{j=1,\ldots,n}\{\|a_j\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}(\Re(a_j)), \mathsf{L}_{\mathfrak{A}}(\Im(a_j))\}$.

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Theorem (Heisenberg Modules (L., 17), informal)

If $(\theta_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}\setminus\mathbb{Q}$ converging to an irrational number θ, and $p, q \in \mathbb{Z}\setminus\{0\}$, then:

$$\lim_{n\to\infty} \Lambda^{\mathsf{mod}}\left(\mathscr{H}_{\theta_n}^{p,q}, \mathscr{H}_{\theta}^{p,q}\right) = 0$$

where $\mathscr{H}_{\vartheta}^{p,q}$ is the (non-free, projective, f.g.) module over $\mathscr{A}_\vartheta$ of trace $q\vartheta - p$ in $K_0(\mathscr{A}_\vartheta)$ for all irrational number ϑ.

Theorem (L., 17)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and p,q fixed. If for all $\theta \in \mathbb{R}$, and $a \in \mathscr{A}_\theta$:

$$\mathsf{L}_\theta(a) = \sup\left\{ \frac{\left\| \beta_\theta^{\exp(ix),\exp(iy)} a - a \right\|_{\mathscr{A}_\theta}}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where β_θ is the dual action, and for all $\xi \in \mathscr{H}_\theta^{p,q}$ we set:

$$\mathsf{D}_\theta^{p,q}(\xi) = \sup\left\{ \frac{\left\| \alpha_{\eth}^{x,y,\frac{xy}{2}} \xi - \xi \right\|_{\mathscr{H}_\theta^{p,q}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where $\eth = \theta - p/q$ and $\alpha_\eth$ is the action of the Heisenberg group, then:

$$\lim_{\vartheta\to\theta} \Lambda^{\mathsf{mod}}\left(\left(\mathscr{H}_\vartheta^{p,q}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\vartheta^{p,q}}, \mathsf{D}_\vartheta^{p,q}, \mathscr{A}_\vartheta, \mathsf{L}_\vartheta\right), \left(\mathscr{H}_\theta^{p,q}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q}}, \mathsf{D}_\theta^{p,q}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)\right) = 0.$$

Thank you!

- *The Quantum Gromov-Hausdorff Propinquity*, F. Latrémolière, *Transactions of the AMS* **368** (2016) 1, pp. 365–411, ArXiv: 1302.4058
- *Convergence of Fuzzy Tori and Quantum Tori for the quantum Gromov-Hausdorff Propinquity: an explicit Approach*, F. Latrémolière, *Münster Journal of Mathematics* **8** (2015) 1, pp. 57–98, ArXiv: 1312.0069
- *The Dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Journal de Mathématiques Pures et Appliquées* **103** (2015) 2, pp. 303–351, ArXiv: 1311.0104
- *A compactness theorem for the dual Gromov-Hausdorff Propinquity*, F. Latrémolière, Accepted in *Indiana University Mathematics Journal* (2015), 40 pages, Arxiv: 1501.06121
- *Quantum Ultrametrics on AF algebras*, K. Aguilar and F. Latrémolière, *Studia Math.* **231** (2015) 2, pp. 149–193, ArXiv: 1511.07114
- *Equivalence of Quantum Metrics with a common domain*, F. Latrémolière, *Journal of Math. Anal. and Appl.* **443** (2016) 2. 1179–1195, ArXiv: 1604.00755
- *The Modular Gromov-Hausdorff Propinquity*, F. Latrémolière, Submitted (2017), 130 pages, ArXiv: 1608.04881, 1703.07073