Convergence of Quantum Metric Spaces

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Noncommutative Metric Geometry

Founding Allegory of Noncommutative Metric Geometry

Noncommutative *metric* geometry is the study of noncommutative generalizations of algebras of *Lipschitz* functions over *metric* spaces.

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- Pioneered by *Rieffel (1998–)*, inspired by *Connes (1989)*.
- Motivated by mathematical physics, addresses problems such as:
  - Can we approximate quantum spaces with finite dimensional algebras?
  - Are certain functions from a topological space to quantum spaces continuous? Lipschitz?
  - Are certain functions from a topological space to modules over quantum spaces continuous?

Structure of the talk

1. Compact Quantum Metric Spaces
2. Convergence of quasi-Leibniz quantum compact metric space
3. The Modular Propinquity

GPS

1 Compact Quantum Metric Spaces

2 Convergence of quasi-Leibniz quantum compact metric space

3 The Modular Propinquity

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\mathsf{L}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{d}(x,y)} : x, y \in X, x \neq y \right\}$$

for all $f \in C(X)$ (allowing ∞).

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The *Monge-Kantorovich metric* on $\mathscr{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

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The Gelfand map $x \in (X, \mathsf{m}) \mapsto \delta_x \in (\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}})$ is an isometry.

Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

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We call L an *L-seminorm*.

Lipschitz morphisms and Quantum Isometries

Theorem-Definition (Lipschitz Morphisms)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *k-Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that any of the following equivalent statement holds:

1. $\varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$ is a *k-Lipschitz map from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ to $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$*,
2. (Rieffel, 00) $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant k \mathsf{L}_{\mathfrak{A}}$,
3. (L., 16) $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

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Definition (Rieffel (98), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that:

$$\forall b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) \quad \mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.$$

A *full quantum isometry* π is a *-isomorphism such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

Ergodic actions of compact metric groups

Theorem (Rieffel, 98)

Let G be a *compact group* endowed with a *continuous length function* ℓ. Let α be an *action* of G on some *unital C*-algebra* $\mathfrak{A}$. Set:

$$\forall a \in \mathfrak{A} \quad \mathsf{L}(a) = \sup\left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1_G\} \right\}.$$

$(\mathfrak{A}, \mathsf{L})$ is a *Leibniz quantum compact metric space* if and only if $\left\{a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a\right\} = \mathbb{C}1_{\mathfrak{A}}$.

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Example: Quantum tori

- $G = \mathbb{T}^d$,
- $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$ (universal for $U_j U_k = \sigma(j,k) U_{j+k}$),
- α : dual action ($\alpha^z U_j = z^j U_j$).
- Associated with a differential calculus when ℓ from invariant Riemannian metric.

Spectral triples and quantum metrics

Theorem (L., 15)

Let α be an *ergodic action* of a *compact Lie group G* on a unital C*-algebra $\mathfrak{A}$. Let π be the GNS rep from the invariant tracial state of $\mathfrak{A}$. Let $(X_1, \ldots, X_n)$ be an orthonormal basis for the *Lie algebra $\mathfrak{g}$* of G equipped with an inner product, and let:

$$\partial_j : a \in \mathfrak{A}^1 \mapsto \lim_{t \to 0} \frac{\alpha^{\exp(tX_j)} a - a}{t} \text{ for } j \in \{1, \ldots, n\}.$$

For $H = \begin{pmatrix} h_{11} & \ldots & h_{1n} \\ \vdots & & \vdots \\ h_{n1} & \ldots & h_{nn} \end{pmatrix} \in M_n(\pi(\mathfrak{A})')$, we set:

$$D_H = \sum_{j=1}^{n} \sum_{k=1}^{n} h_{jk} \partial_k \otimes \mathrm{Clifford}(X_j)$$

$\left(\mathfrak{A}, \left|\left|\left| [D_H, \pi^{\oplus n}(\cdot)] \right|\right|\right|\right)$ is a *Leibniz quantum compact metric space*.

AF algebras with tracial state

Theorem (Aguilar, L., 15)

- Let $\mathfrak{A} = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$ be an *AF-algebra with a faithful tracial state* τ and where $\dim \mathfrak{A}_n < \infty$ for all $n \in \mathbb{N}$.
- For all $n \in \mathbb{N}$ let $\mathbb{E}_n : \mathfrak{A} \to \mathfrak{A}_n$ be the unique conditional expectation with $\tau \circ \mathbb{E}_n = \tau$.
- Let $(\beta_n)_{n\in\mathbb{N}}$ in $(0,\infty)^{\mathbb{N}}$, with limit 0.

If, for all $a \in \mathfrak{sa}(\mathfrak{A})$, we set:

$$\mathsf{L}(a) = \sup\left\{ \frac{\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}}}{\beta_n} : n \in \mathbb{N} \right\}$$

then $(\mathfrak{A}, \mathsf{L})$ is a $(2,0)$–quasi-Leibniz quantum compact metric space.

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The quasi-Leibniz condition here is:

$$\forall a, b \in \mathfrak{A} \quad \max\{\mathsf{L}(a\circ b), \mathsf{L}(\{a,b\})\} \leqslant 2(\mathsf{L}_{\mathfrak{A}}(a)\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}}\mathsf{L}_{\mathfrak{A}}(b)).$$

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This construction recovers the usual ultrametrics of the Cantor set.

Some Other Examples

1. Hyperbolic group C*-algebras (Rieffel, Ozawa, 05),
2. Nilpotent group C*-algebras (Christ, Rieffel, 16),
3. Connes-Landi spheres (Li, 03)
4. Conformal deformations of quantum metric spaces from spectral triples (L., 15)
5. Group C*-algebras for groups with rapid decay (Antonescu, Christensen, 2004)
6. Noncommutative Solenoids (L., Packer, 16)
7. Certain C*-crossed-products (J. Bellissard, M. Marcolli, Reihani, 10), (involves my work on locally compact quantum metric space).

GPS

1. Compact Quantum Metric Spaces
2. Convergence of quasi-Leibniz quantum compact metric space
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The Gromov-Hausdorff Distance

Definition

For any two compact metric spaces (X, m_X) and (Y, m_Y), we define $\mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y)$ as:

$$\left\{ (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \,\middle|\, \begin{array}{l} (Z, \mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\}$$

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Notation

The *Hausdorff distance* on the compact subsets of a metric space (X, m) is denoted by $\mathsf{Haus}_{\mathsf{m}}$.

Definition (Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X, m_X) and (Y, m_Y) is:

$$\inf \left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X), \iota_Y(Y)) : (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \in \mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y) \right\}.$$

A noncommutative Gromov-Hausdorff distance

Problem

How to generalize Gromov's construction to quasi-Leibniz quantum compact metric spaces ?

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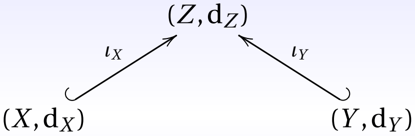

Figure: Isometric Embeddings

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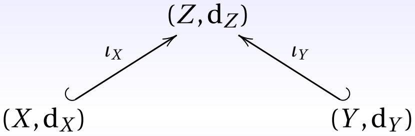

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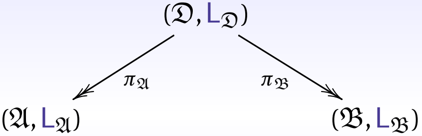

Figure: A tunnel

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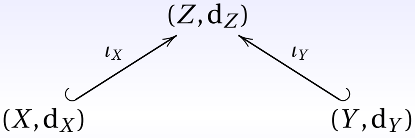

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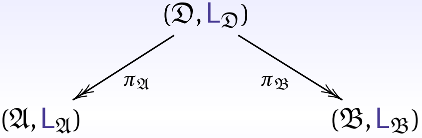

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Choices

- What class $(\mathfrak{D}, \mathsf{L}_{\mathfrak{B}})$ should belong to? *Should we assume a form of quasi-Leibniz inequality?*
- What kind of morphisms $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ should we choose?
- How do we quantify a bridge?

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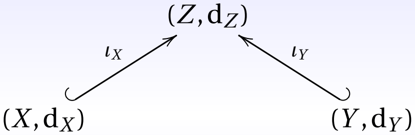

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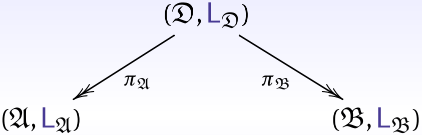

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Previous problems

- *Coincidence problem:* distance zero may not imply *-isomorphism (Rieffel, Wu)
- *Triangle Inequality:* working with quasi-Leibniz $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$ meant abandoning the triangle inequality (Kerr, Rieffel)

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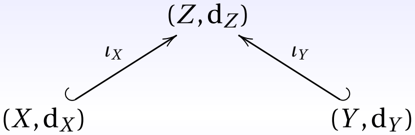

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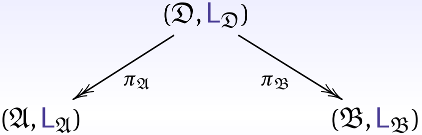

Figure: A tunnel

Previous problem root cause

Noncommutative Gromov-Hausdorff distances construction went out of the category of quasi-Leibniz quantum compact metric spaces. Thus, no easy applications to modules, morphisms, …

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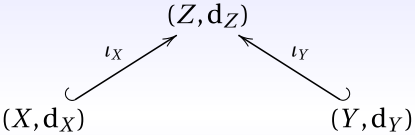

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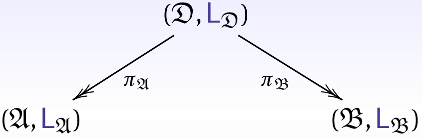

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Solution (L., 13)

- We choose isometric embeddings which are **-morphisms*.
- We restrict embeddings to *quasi-Leibniz quantum compact metric spaces* or even more specific if desired.

The Dual Gromov-Hausdorff Propinquity

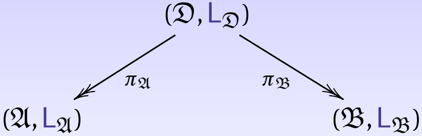

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

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$$\begin{array}{ccc} & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) & \\ \swarrow^{\pi_{\mathfrak{A}}} & & \searrow^{\pi_{\mathfrak{B}}} \\ (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \end{array}$$

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

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The *extent* of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{A}))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}(\mathscr{S}(\mathfrak{B}))\right)\right\}.$$

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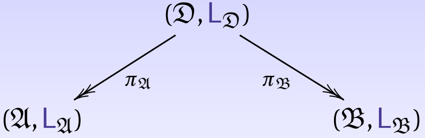

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The *dual propinquity* $\Lambda_F^{*}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

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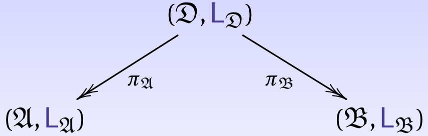

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Theorem (L., 13)

The dual propinquity is a *complete metric* up to *full quantum isometry*, which induces the same topology on classical compact metric spaces as the Gromov-Hausdorff distance.

Quantum Tori and the quantum propinquity

Endow $\mathbb{T}^d$ with a continuous length function ℓ. Let α be the dual action of $\widehat{\mathbb{Z}_k} \subseteq \mathbb{T}^d$ on $C^*(\mathbb{Z}_k^d, \sigma)$, and set for $a \in \mathfrak{sa}\left(\mathbb{C}^*(\mathbb{Z}_k^d, \sigma)\right)$:

$$\mathsf{L}_\sigma(a) = \sup\left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in \widehat{\mathbb{Z}_k^d} \setminus \{1\} \right\}.$$

L_σ is an L-seminorm (Rieffel, 98).

Theorem (Latrémolière, 2013)

Let $d \in \mathbb{N} \setminus \{0,1\}$, σ a multiplier of $\mathbb{Z}^d$. If for each $n \in \mathbb{N}$, we let $k_n \in \overline{\mathbb{N}}^d$ and σ_n be a multiplier of $\mathbb{Z}_k^d = \mathbb{Z}^d / k_n \mathbb{Z}^d$ such that:

1. $\lim_{n\to\infty} k_n = (\infty, \ldots, \infty)$,
2. the unique lifts of σ_n to $\mathbb{Z}^d$ as multipliers converge pointwise to σ,

then $\lim_{n\to\infty} \Lambda\left(\left(C^*\left(\mathbb{Z}^d, \sigma\right), \mathsf{L}_\sigma\right), \left(C^*\left(\mathbb{Z}_{k_n}^d, \sigma_n\right), \mathsf{L}_{\sigma_n}\right)\right) = 0.$

Curved Quantum Tori

Theorem (L., 15)

- Let σ be a multiplier of $\mathbb{Z}^d$, and $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$.
- Identify $\mathfrak{A}$ with its image by the regular representation acting on $L^2(\mathfrak{A}, \tau)$.
- Define the length function $\ell : K \in \mathrm{GL}_n(\mathfrak{A}') \mapsto |||1 - K|||_{L^2(\mathfrak{A},\tau)}$.

If $H \in \mathrm{GL}_n(\mathfrak{A}')$ then:

$$\lim_{\substack{G \to H \\ G \in \mathfrak{GL}_n(\mathfrak{A}')}} \Lambda((\mathfrak{A}, \mathsf{L}_G), (\mathfrak{A}, \mathsf{L}_H)) = 0$$

where $\mathsf{L}_H = |||[D_H, \cdot]|||$ with $D_H = \sum_{j=1}^{n} \sum_{k=1}^{n} H_{jk} \partial_k \otimes X_j$ for a fixed orthonormal basis $(X_1, \ldots, X_d)$ of $\mathbb{R}^d$ seen as the Lie algebra of $\mathbb{T}^d$.

Effros-Shen AF algebras

Theorem (Aguilar, L., 15)

- For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $\theta = \lim_{n\to\infty} \frac{p_n^\theta}{q_n^\theta}$ with $\frac{p_n^\theta}{q_n^\theta} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}$ for $a_1, \ldots \in \mathbb{N}$.
- Set $\mathfrak{AF}_\theta = \varinjlim_{n\to\infty} \left(\mathfrak{M}_{q_n} \oplus \mathfrak{M}_{q_{n-1}}, \psi_{n,\theta}\right)$ where $\psi_{n,\theta}$ involves a_{n+1}.
- For all $n \in \mathbb{N}$, let $\beta_n = \frac{1}{q_n^2 + q_{n-1}^2}$ and L_θ the L-seminorm for this data.

For all $\theta \in \mathbb{R} \setminus \mathbb{Q}$, we have:

$$\lim_{\substack{\vartheta\to\theta \\ \vartheta\notin\mathbb{Q}}} \Lambda\left((\mathfrak{AF}_\vartheta, \mathsf{L}_\vartheta), (\mathfrak{AF}_\theta, \mathsf{L}_\theta)\right) = 0.$$

Other examples

1. Conformal perturbations of quantum metrics (L., 15)
2. AF algebras as limits of their inductive sequence in a *metric* sense; UHF and Effros-Shen algebras form continuous families (Aguilar and L., 15),
3. Spheres as limits of full matrix algebras (Rieffel, 15)
4. Nuclear quasi-diagonal quasi-Leibniz quantum compact metric spaces have finite dim approximations (L., 15),
5. There exists an analogue of Gromov's compactness theorem (L., 15)
6. Noncommutative solenoids form a continuous family and have approximations by quantum tori (L. and Packer, 16)
7. Closed balls for the noncommutative Lipschitz distance are totally bounded for Λ (L., 16)

GPS

1 Compact Quantum Metric Spaces

2 Convergence of quasi-Leibniz quantum compact metric space

3 The Modular Propinquity

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Example: Classical picture

Hermitian bundles over compact connected Riemannian manifolds.

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Example: Free modules

Given $(\mathfrak{A}, \mathsf{L})$, we set $\langle(a_1,\ldots,a_d),(b_1,\ldots,b_d)\rangle_d = \sum_{j=1}^{d} a_j b_j^*$ and $\mathsf{L}_d(a_1,\ldots,a_d) = \max\left\{\mathsf{L}(\Re a_j), \mathsf{L}(\Im a_j) : j \in \{1,\ldots,d\}\right\}$. Let $\mathsf{D} = \max\{\|\cdot\|_d, \mathsf{L}_d\}$. Then $(\mathfrak{A}^d, \langle,\rangle_d, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is a metrized quantum vector bundle.

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Example: Heisenberg Modules

Heisenberg modules and their natural *connection*, as build by Connes (81), are (non-free, finitely generated, projective) metrized quantum vector bundles.

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Theorem-Definition (The Modular Propinquity (L.,16))

There exists a *distance Λ^{mod}, up to full isometry,* on the class of metrized quantum vector bundles, whose restriction to quasi-Leibniz quantum compact metric spaces, is equivalent to the quantum propinquity.

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Theorem (Free Modules (L., 17))

If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are quasi-Leibniz quantum compact metric space then:

$$\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant$$
$$\Lambda^{\mathsf{mod}}((\mathfrak{A}^n, \mathsf{D}^n_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}^n, \mathsf{D}^n_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant 2n\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$$

where $\mathsf{D}^n_{\mathfrak{A}}(a_1, \ldots, a_n) = \max_{j=1,\ldots,n}\{\|a_j\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}(\Re(a_j)), \mathsf{L}_{\mathfrak{A}}(\Im(a_j))\}$.

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Theorem (Heisenberg Modules (L., 17), informal)

If $(\theta_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}\setminus\mathbb{Q}$ converging to an irrational number θ, and $p,q\in\mathbb{Z}\setminus\{0\}$, then:

$$\lim_{n\to\infty}\Lambda^{\mathsf{mod}}\left(\mathscr{H}_{\theta_n}^{p,q},\mathscr{H}_{\theta}^{p,q}\right)=0$$

where $\mathscr{H}_{\vartheta}^{p,q}$ is the (non-free, projective, f.g.) module over $\mathscr{A}_\vartheta$ of trace $q\vartheta-p$ in $K_0(\mathscr{A}_\vartheta)$ for all irrational number ϑ.

Theorem (L., 17)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and p,q fixed. If for all $\theta\in\mathbb{R}$, and $a\in\mathscr{A}_\theta$:

$$\mathsf{L}_\theta(a)=\sup\left\{\frac{\left\|\beta_\theta^{\exp(ix),\exp(iy)}a-a\right\|_{\mathscr{A}_\theta}}{\|(x,y)\|}:(x,y)\in\mathbb{R}^2\setminus\{0\}\right\}$$

where β_θ is the dual action, and for all $\xi\in\mathscr{H}_\theta^{p,q}$ we set:

$$\mathsf{D}_\theta^{p,q}(\xi)=\sup\left\{\frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}}\xi-\xi\right\|_{\mathscr{H}_\theta^{p,q}}}{2\pi|\eth|\|(x,y)\|}:(x,y)\in\mathbb{R}^2\setminus\{0\}\right\}$$

where $\eth=\theta-p/q$ and $\alpha_\eth$ is the action of the Heisenberg group, then:

$$\lim_{\vartheta\to\theta}\Lambda^{\mathsf{mod}}\left(\left(\mathscr{H}_\vartheta^{p,q},\langle\cdot,\cdot\rangle_{\mathscr{H}_\vartheta^{p,q}},\mathsf{D}_\vartheta^{p,q},\mathscr{A}_\vartheta,\mathsf{L}_\vartheta\right),\left(\mathscr{H}_\theta^{p,q},\langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q}},\mathsf{D}_\theta^{p,q},\mathscr{A}_\theta,\mathsf{L}_\theta\right)\right)=0.$$

Thank you!

- *The Quantum Gromov-Hausdorff Propinquity*, F. Latrémolière, *Transactions of the AMS* **368** (2016) 1, pp. 365–411, ArXiv: 1302.4058
- *Convergence of Fuzzy Tori and Quantum Tori for the quantum Gromov-Hausdorff Propinquity: an explicit Approach*, F. Latrémolière, *Münster Journal of Mathematics* **8** (2015) 1, pp. 57–98, ArXiv: 1312.0069
- *The Dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Journal de Mathématiques Pures et Appliquées* **103** (2015) 2, pp. 303–351, ArXiv: 1311.0104
- *A compactness theorem for the dual Gromov-Hausdorff Propinquity*, F. Latrémolière, Accepted in *Indiana University Mathematics Journal* (2015), 40 pages, Arxiv: 1501.06121
- *Quantum Ultrametrics on AF algebras*, K. Aguilar and F. Latrémolière, *Studia Math.* **231** (2015) 2, pp. 149–193, ArXiv: 1511.07114
- *The Modular Gromov-Hausdorff Propinquity*, F. Latrémolière, Submitted (2015), 130 pages, ArXiv: 1608.04881, 1703.07073

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A metrized quantum vector bundle $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is given by:

1. $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space,
2. $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ is a left Hilbert module over $\mathfrak{A}$,
3. D is a norm on a dense subspace of $\mathscr{M}$ such that:
    1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
    2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
    3. $\mathsf{D}(a\omega) \leqslant G(\|a\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{D}(\omega))$,
    4. $\mathsf{L}(\langle\omega,\eta\rangle_{\mathscr{M}}) \leqslant H(\mathsf{D}(\omega), \mathsf{D}(\eta))$.

Example: Classical picture

Hermitian bundles over compact connected Riemannian manifolds.

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Example: Free modules

Given $(\mathfrak{A}, \mathsf{L})$, we set $\langle(a_1,\ldots,a_d),(b_1,\ldots,b_d)\rangle_d = \sum_{j=1}^d a_j b_j^*$ and $\mathsf{L}_d(a_1,\ldots,a_d) = \max\left\{\mathsf{L}(\Re a_j), \mathsf{L}(\Im a_j) : j \in \{1,\ldots,d\}\right\}$. Let $\mathsf{D} = \max\{\|\cdot\|_d, \mathsf{L}_d\}$. Then $(\mathfrak{A}^d, \langle,\rangle_d, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is a metrized quantum vector bundle.

The Heisenberg Modules (Connes, 81; Rieffel)

Fix $\theta \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $d \in \mathbb{N} \setminus \{0\}$ such that $\eth = \theta - \frac{p}{q} \neq 0$.

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1. Start with a representation of $\left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R}^3 \right\}$ on $L^2(\mathbb{R})$:

$$\alpha_{\eth}^{x,y,t} \xi(s) = \exp(i\pi(t + 2xs))\xi(s + \eth y).$$

Promote it to $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.

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Promote it to $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.
2. Let $W_1, W_2 \in U(d)$ with $W_1 W_2 = e^{2i\pi p/q} W_2 W_1$ and $W_1^n = W_2^n = 1$. We get a $\mathscr{A}_\theta = C^*(u_\theta, v_\theta)$-module with:
$$(u_\theta^n v_\theta^m)\xi = W_1^n W_2^m \alpha_{\eth}^{n,m,0} \xi.$$

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3. For Schwarz functions ξ, ω, set:
$$\langle \xi, \omega \rangle_{\mathscr{H}_\theta^{p,q,d}} = \sum_{n,m \in \mathbb{Z}} \langle u_\theta^n v_\theta^m \xi, \omega \rangle_{L^2(\mathbb{R}, \mathbb{C}^d)} u_\theta^n v_\theta^m;$$
complete space of Schwarz functions to the *Heisenberg module* $\mathscr{H}_\theta^{p,q,d}$.

The D-norm

The action of $\mathbb{H}_3$ on Heisenberg modules define a connection. As with the quantum tori, though using a different proof:

Definition (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

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Theorem (L., 16)

$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

Seminorms from Differential Calculi

- Let α be a nice action of a *Lie group* G on a Banach space $\mathscr{E}$.
- Let $\mathfrak{h}$ a subspace of the *Lie algebra* $\mathfrak{g}$ of G and $\|\cdot\|$ be a norm on $\mathfrak{h}$.

For e in a dense subspace of $\mathscr{E}$, the following limits exist:

$$\nabla e : X \in \mathfrak{h} \mapsto \nabla_X e = X(e) = \lim_{t \to 0} \frac{\alpha^{\exp(tX)} e - e}{t}.$$

We denote by $|||\nabla e|||$ the norm of $X \in \mathfrak{h} \mapsto \nabla_X e$.

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$$\begin{aligned} |||\nabla e||| &= \sup\left\{ \frac{\left\|\alpha^{\exp(X)} e - e\right\|_{\mathscr{E}}}{\|X\|} : X \in \mathfrak{h} \setminus \{0\} \right\} \\ &= \limsup_{\|X\|\to 0} \frac{\left\|\alpha^{\exp(X)} e - e\right\|_{\mathscr{E}}}{\|X\|}. \end{aligned}$$

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An idea

We could use such differential calculi and associated norms to build quantum metrics.

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Quantum Tori

$G = \mathbb{T}^d$, $\mathscr{E} = \mathscr{A}_\theta$, α is dual action, and $\mathfrak{h} = \mathbb{R}^d$: we get back L_θ for ℓ the path metric length.

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Heisenberg Modules

$G = \mathbb{H}_3$, $\mathscr{E} = \mathscr{H}_\theta^{p,q,d}$, and $\mathfrak{h} = \operatorname{span}\{P, Q\}$. We get the Yang-Mills connection and our D-norm.

Bridges

Theorem (L. (13), Informal)

For any *bridge* $\gamma = (\mathfrak{E}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ where $\mathfrak{E}$ is a unital C*-algebra, $x \in \mathfrak{E}$ and $\pi_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{E}$, $\pi_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{E}$ are unital *-monomorphisms, there exists $\lambda(\gamma) > 0$ such that:

$$(\mathfrak{A} \oplus \mathfrak{B}, (a,b) \mapsto a, (a,b) \mapsto b, \\ (a,b) \mapsto \max\left\{ \mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), \frac{1}{\lambda(\gamma)} \|\pi_{\mathfrak{A}}(a)x - x\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}} \right\}$$

is a tunnel of extend at most $2\lambda(\gamma)$.

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is a tunnel of extend at most $2\lambda(\gamma)$.

We may use the *length* of a bridge to construct a distance on quasi-Leibniz quantum compact metric spaces, the *quantum propinquity* Λ, which dominates the dual propinquity, and induces the same topology on classical compact metric spaces.
All proofs of convergence to date for the dual propinquity are in fact done for the stronger quantum propinquity.

Bridges for modules

Fix $\Omega_{\mathfrak{A}} = (\mathscr{M}_{\mathfrak{A}}, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $\Omega_{\mathfrak{B}} = (\mathscr{M}_{\mathfrak{B}}, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two metrized quantum vector bundles.

Definition (L., 16)

A *modular bridge* $\left(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j)_{j\in J}, (\eta_j)_{j\in J}\right)$ is a bridge $(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ and two families $(\omega_j)_{j\in J} \in \mathscr{M}_{\mathfrak{A}}$, $(\eta_j)_{j\in J} \in \mathscr{M}_{\mathfrak{B}}$ with $\mathsf{D}_{\mathfrak{A}}(\omega_j), \mathsf{D}_{\mathfrak{B}}(\eta_j) \leqslant 1$ for all $j \in J$.

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Definition (L., 16)

The *length* of a modular bridge is the maximum of the length of its basic bridge, and the sum of:

1. the maximum of $\mathsf{Haus}_k(\{\omega_j : j \in J\}, \{\omega : \mathsf{D}_{\mathfrak{A}}(\omega) \leqslant 1)$ and its counterpart in $\Omega_{\mathfrak{B}}$, where:

$$k(\omega,\xi) = \sup\left\{\|\langle\omega,\eta\rangle_{\mathfrak{A}} - \langle\xi,\eta\rangle_{\mathfrak{A}}\|_{\mathfrak{A}} : \mathsf{D}_{\mathfrak{A}}(\eta) \leqslant 1\right\},$$

2. $\max\left\{\|\pi_{\mathfrak{A}}(\langle\omega_j,\omega_k\rangle_{\mathfrak{A}})x - x\pi_{\mathfrak{B}}(\langle\eta_j,\eta_k\rangle_{\mathfrak{B}})\|_{\mathfrak{D}} : j \in J\right\}$.

The modular propinquity

Definition (L., 16)

The *modular propinquity* is the largest pseudo-metric Λ^{mod} such that $\Lambda^{\mathsf{mod}}(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}) \leqslant \lambda\left(\gamma\right)$ for any modular γ from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$.

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Theorem (L., 16)

The *modular propinquity* is a metric on metrized quantum vector bundles up to full quantum isometry.

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Theorem (L., 16)

The *modular propinquity* is a metric on metrized quantum vector bundles up to full quantum isometry.

Theorem (Free modules; L., 16)

If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are quasi-Leibniz quantum compact metric space then:

$$\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant$$
$$\Lambda^{\mathsf{mod}}((\mathfrak{A}^n, \mathsf{D}^n_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}^n, \mathsf{D}^n_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant 2n\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$$

where $\mathsf{D}^n_{\mathfrak{A}}(a_1, \ldots, a_n) = \max_{j=1,\ldots,n}\{\|a_j\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}(\Re(a_j)), \mathsf{L}_{\mathfrak{A}}(\Im(a_j))\}$.

Theorem (L., 16)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and p, q, d fixed. If for all $\theta \in \mathbb{R}$, and $a \in \mathscr{A}_\theta$:

$$\mathsf{L}_\theta(a) = \sup\left\{ \frac{\left\|\beta_\theta^{\exp(ix),\exp(iy)} a - a\right\|_{\mathscr{A}_\theta}}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where β_θ is the dual action, and for all $\xi \in \mathscr{H}_\theta^{p,q,d}$ we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}} \xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where $\eth = \theta - p/q$, then:

$$\lim_{\vartheta \to \theta} \Lambda^{\mathsf{mod}}\left(\left(\mathscr{H}_\vartheta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\vartheta^{p,q,d}}, \mathsf{D}_\vartheta^{p,q,d}, \mathscr{A}_\vartheta, \mathsf{L}_\vartheta\right), \left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)\right) = 0.$$