Compact quantum surfaces of any genus

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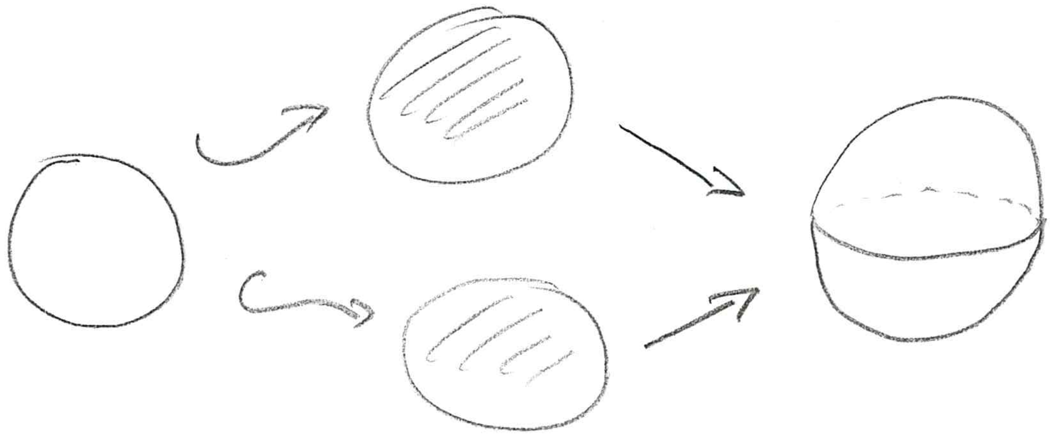
Fibre product construction of the Podleś spheres

Classical 2-spheres



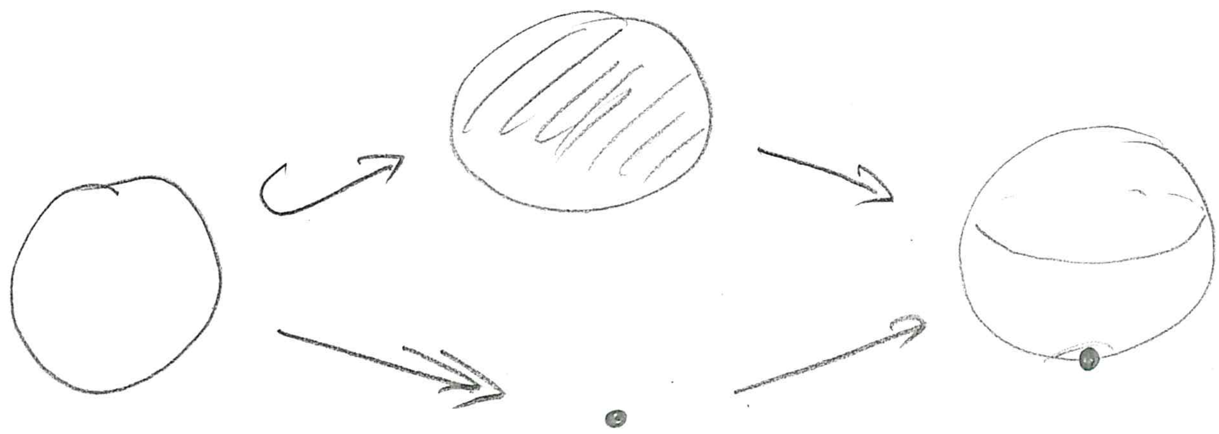
Pullback on functions:

$$\begin{tikzcd}
 & C(\overline{D}) \arrow[dl, twoheadrightarrow] & \\
C(S^1) & & C(S^2) \arrow[ul] \arrow[dl] \\
 & C(\overline{D}) \arrow[ul, twoheadrightarrow] &
\end{tikzcd}$$

$\overline{D} := \{ z \in \mathbb{C} : |z| \leq 1 \}$

$D : \{ z \in \mathbb{C} : |z| < 1 \}$

1 - surjective pull-back diagram



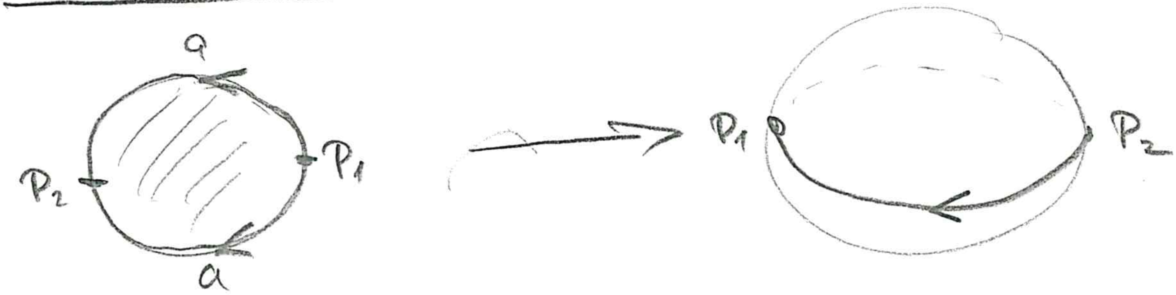
Pullback on functions

$$
\begin{tikzcd}
& C(\bar{D}) \arrow[dl] & \\
C(S^1) & & C(S^2) \arrow[ul] \arrow[dl] \\
& \mathbb{C} \arrow[ul] &
\end{tikzcd}
$$

$\Rightarrow$ 1-point compactification of D :

$$C(S^2) = C_0(D) \oplus \mathbb{C}\mathbb{1}$$

Alternatively



$$C(S^2) = \{ f \in C(\bar{D}) : f(u) = f(\bar{u}) \ \forall u \in$$

<u>Deformations</u>: Replacing $C(\overline{\mathbb{D}})$ by the Toeplitz algebra $\mathcal{T}$

<u>C^*-algebra extension</u>

$$0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(\mathbb{S}^1) \to$$

<u>Interpretation</u>: $\mathcal{T}$ = quantum disc

$$\begin{array}{ccccccc} 0 & \to & C_0(D_q) & \to & C(\overline{D}_q) & \xrightarrow{\sigma} & C(\mathbb{S}^1) \to \\ & & \| & & \| & & \\ & & \mathcal{K}(\ell_2(\mathbb{N})) & & \mathcal{T} & & \end{array}$$

The surjective symbol map

$$C(\overline{D}_q) \overset{\sigma}{\twoheadrightarrow} C(\mathbb{S}^1)$$

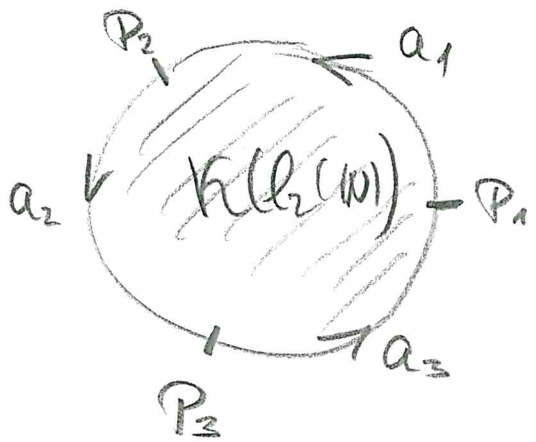
corresponds to

$$\overline{D}_q \hookleftarrow \mathbb{S}^1$$

(embedding)

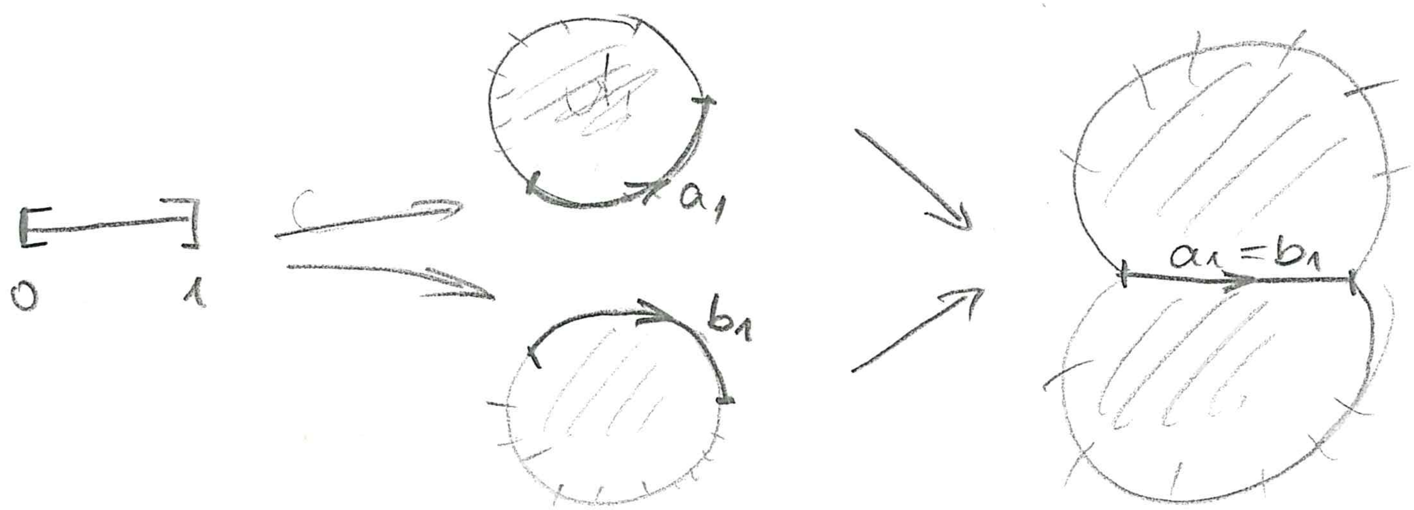
Noncommutative simplicial complexes

Standard 2-simplex



$= \mathcal{T}$

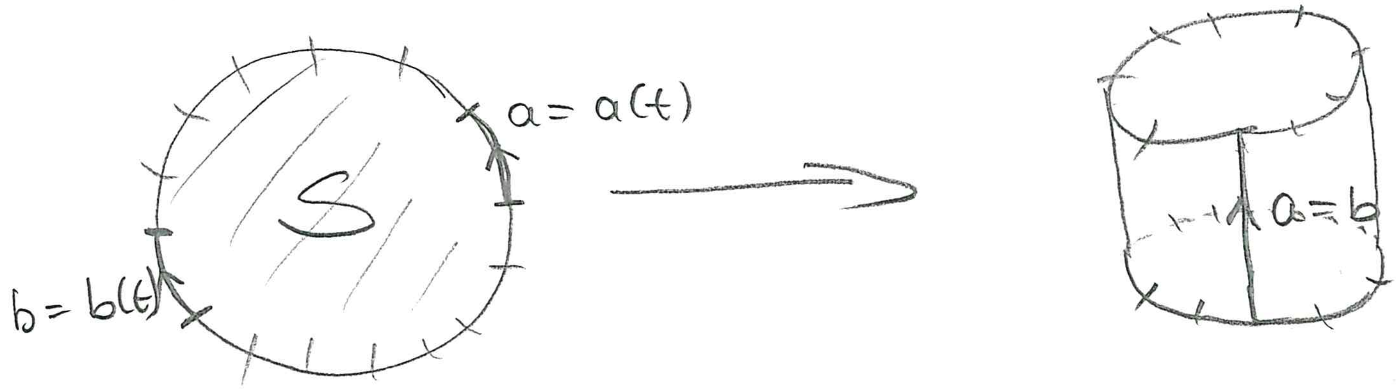
Gluing two simplexes



C*-algebras:

$$\begin{tikzcd}
 & A_1 \oplus_{C([0,1])} A_2 \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
A_1 \arrow[dr] & & A_2 \arrow[dl] \\
 & C([0,1]) &
\end{tikzcd}$$

Identifying edges



C^*-algebras:

$$\sigma : \mathcal{A} \twoheadrightarrow C(\partial S)$$

$$\mathcal{B} := \{ f \in \mathcal{A} : \sigma(f)(a(t)) = \sigma(f)(b(t)) \ \forall t \in [0$$

$\subset \mathcal{A}$ subalgebra

Higher dimensions

C^*-algebra extensions

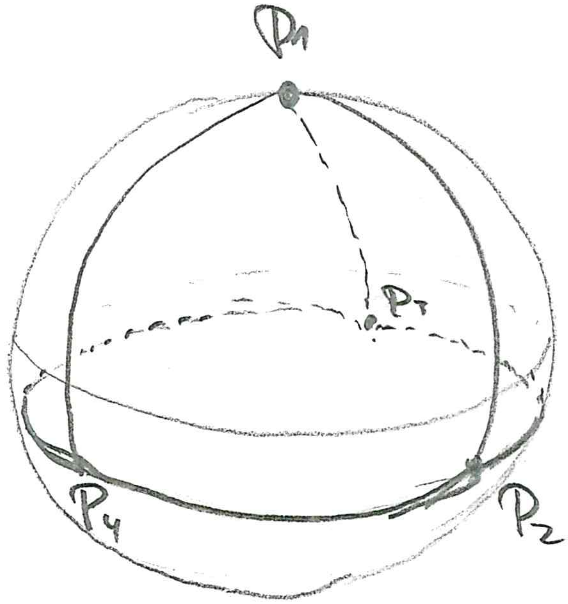
$$0 \to \mathcal{K} \to \mathcal{A} \xrightarrow{\sigma} C(S^n) \to 0$$

n - simplex:

generated by $(n+1)$ - "generic"

Points.

Example: 3 - simplex



Examples for C^*-algebras in higher dimensions

$\mathcal{T}_1 := \mathcal{T}$ – Toeplitz algebra

$\mathcal{T}_n := \Sigma \mathcal{T}_{n-1}$ = unreduced suspension of $\mathcal{T}_{n-1}$

$$\Rightarrow \partial \mathcal{T}_n = \Sigma \, \partial \mathcal{T}_{n-1}$$
$$= \Sigma(\dots (\Sigma \partial \mathcal{T}) \dots)$$
$$= \Sigma(\dots (\Sigma C(S^1)) \dots)$$
$$\cong C(S^n)$$

C^*-algebra extension

$$0 \longrightarrow \Sigma_0(\dots (\Sigma_0 K(\ell_2(\mathbb{N}))) \dots) \longrightarrow \mathcal{T}_n \longrightarrow C(S^n) \longrightarrow$$

Σ_0 = reduced suspension

Observation: $K_0(C_0(\mathbb{D})) = K_0(K(\ell_2(\mathbb{N})))$
$K_1(C_0(\mathbb{D})) = K_1(K(\ell_2(\mathbb{N})))$

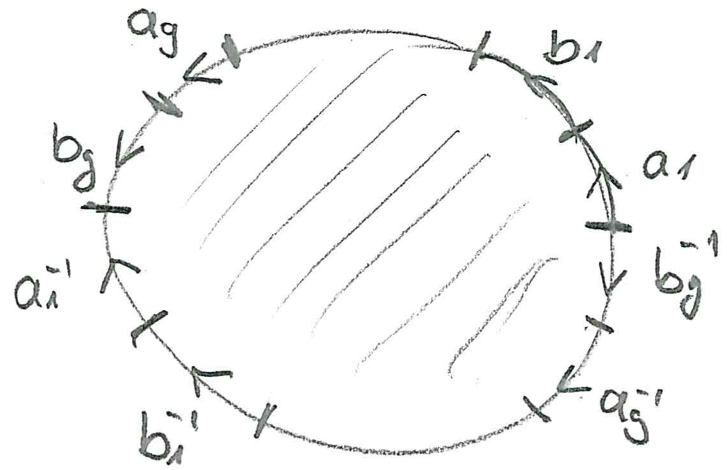
$$\Rightarrow K_*(\underbrace{\Sigma_0(\dots (\Sigma_0}_{n\text{-}1\text{ - times}} K(\ell_2(\mathbb{N}))) \dots)) = K_*(C_0(\mathbb{R}^{2+n}))$$

Future project: Study C^*-algebras of noncommutative simplicial complexes and their K-Theory.

Compact quantum surfaces of any genus

Orientable

Π_g :



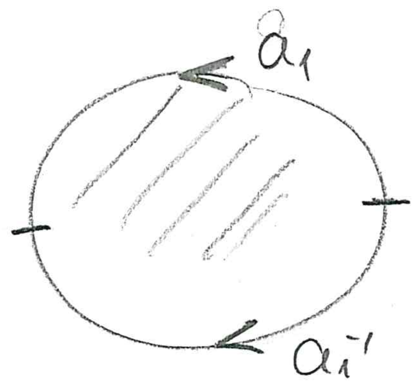
$, g \geq$

quantum

$$C(\Pi_{g,q}) := \{ f \in \mathcal{T} : \sigma(f)(a_i(t)) = \sigma(f)(a_i^{-1}(t)), \quad \sigma(f)(b_i(t)) = \sigma(f)(b_i^{-1}(t)), \quad \forall i = 1, \dots, g \}$$

$$\subset \mathcal{T}$$

S^2 :

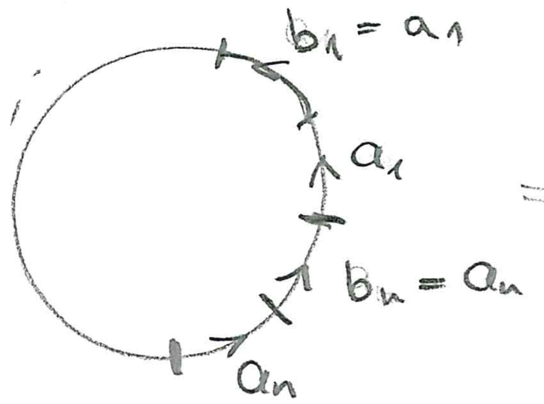


quantum :

$$C(S^2_q) := \{ f \in \mathcal{T} : \sigma(f)(u) = \sigma(f)(\bar{u}) \ \forall u \in S^1 \}$$

Non-orientable

P_n :

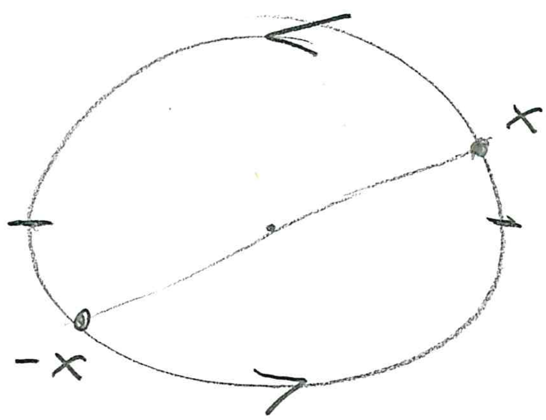


quantum :

$$C(\mathbb{P}_{n,q}) := \{ f \in \mathcal{T} : \sigma(f)(a_i(t)) = \sigma(f)(b_i(t)) \\ \forall i = 1, \dots n$$

Example :

$$C(\mathbb{R}P^2_q) := \{ f \in \mathcal{T} : \sigma(f)(u) = \sigma(f)(-u) \\ \forall u \in S^1 \}$$



K-Theory of quantum surfaces

C^*-algebra extensions

$$0 \longrightarrow C_0(\mathbb{D}) \longrightarrow C_0(\bar{\mathbb{D}}) \longrightarrow C(\mathbb{S}^1) \longrightarrow$$

$$0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(\mathbb{S}^1) \longrightarrow$$

K-groups

$$K_0(C_0(\mathbb{D})) = K_0(\Sigma_0(\Sigma_0\,\mathbb{C})) = K_0(\mathbb{C}) = \mathbb{Z}$$

$$K_0(\mathcal{K}(\ell_2(\mathbb{N}))) = K_0(\mathcal{K}(\ell_2(\mathbb{N})) \otimes \mathbb{C}) = K_0(\mathbb{C}) = \mathbb{Z}$$

$$K_1(C_0(\mathbb{D})) = K_1(\Sigma_0(\Sigma_0(\mathbb{C}))) = K_1(\mathbb{C}) = 0$$

$$K_1(\mathcal{K}(\ell_2(\mathbb{N}))) = K_1(\mathcal{K}(\ell_2(\mathbb{N})) \otimes \mathbb{C}) = K_1(\mathbb{C}) = 0$$

$$K_0(C(\mathbb{S}^1)) = \mathbb{Z}$$

$$K_1(C(\mathbb{S}^1)) = \mathbb{Z}$$

Generators:

$[\mathbb{1}] \in K_0(C(S^1)), \qquad \mathbb{1} \in C(S^1)$

$[u] \in K_1(C(S^1)), \quad u \in C(S^1), \quad u(z) = z$

Bott projection of $C_0(\mathbb{D})$

$$\left[\begin{pmatrix} z \\ \sqrt{1-|z|^2} \end{pmatrix}\left(z^*, \sqrt{1-|z|^2}\right)\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} |z|^2 & \sqrt{1-|z|^2}\, z \\ \sqrt{1-|z|^2}\, \bar{z} & 1-|z|^2 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in K_0(C_0(\mathbb{D}))$$

Elementary projections:

$S : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N}), \quad S e_n = e_{n+1}, \quad$ shift

$[1 - SS^*] \in K_0(K(\ell_2(\mathbb{N})))$

Bott projections from the quantum disc

$z e_n = \sqrt{1 - q^{2(n+1)}}\, S e_n, \quad (1 - zz^*) e_n = q^{2n} e_n$

$= (1 - q^2 z^* z$

$\Rightarrow 1 - zz^* \in K(\ell_2(\mathbb{N}))$

$$\Rightarrow \left[\begin{pmatrix} z^* \\ \sqrt{1-zz^*} \end{pmatrix}\left(z, \sqrt{1-zz^*}\right)\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} z^* z & z^*\sqrt{1-zz^*} \\ \sqrt{1-zz^*}\, z & 1-zz^* \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in K_0(K(\ell_2(\mathbb{N})))$$

$$\operatorname{tr}(z^* z - 1 + 1 - zz^*) = \sum_{n=0}^{\infty}\left(q^{2(n+1)} - q^{2n}\right) = -1$$

6-term exact sequence and index map

$$\begin{tikzcd}
K_0(C_0(\mathbb{D})) \arrow[r] & K_0(C(\bar{\mathbb{D}})) \arrow[r] & K_0(C(S^1)) \arrow[d] \\
K_1(C(S^1)) \arrow[u, "\mathrm{ind}"] & K_1(C(\bar{\mathbb{D}})) \arrow[l] & K_1(C_0(\mathbb{D})) \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "0"'] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[d, "0"] \\
\mathbb{Z} \arrow[u, "\cong"'] & 0 \arrow[l, "0"'] & 0 \arrow[l, "0"']
\end{tikzcd}
\qquad [1] \longmapsto [1]$$

$$\begin{tikzcd}
K_0(K(\ell_2(\mathbb{N}))) \arrow[r] & K_0(\mathcal{T}) \arrow[r] & K_0(C(S^1)) \arrow[d] \\
K_1(C(S^1)) \arrow[u, "\mathrm{ind}"] & K_1(\mathcal{T}) \arrow[l] & K_1(K(\ell_2(\mathbb{N}))) \arrow[l]
\end{tikzcd}$$

$$\mathrm{ind} : K_1(C(S^1)) \longrightarrow \mathbb{Z} \cong K_0(K(\ell_2(\mathbb{N})))$$

$$\qquad [u^n] \longmapsto \mathrm{Ind}(z^n)$$

where $\underset{\text{(lift)}}{\mathcal{T} \ni z^n} \longmapsto u^n \in \underbrace{\mathcal{T} / K(\ell_2(\mathbb{N}))}_{\cong C(S^1)}$

The index map in K-Theory

$$K_1(C(\overline{D})) \longrightarrow K_1(C(S^1)) \xrightarrow{\text{ind}} K_0(C_0(D))$$

Let: $[u_n] \in K_1(C(S^1))$

Lift: $|z|\tilde{u}_n \in C(\overline{D})$

Unitary 2×2 matrix: $V_n = \begin{pmatrix} |z|u_n & -\sqrt{1-|z|^2} \\ \sqrt{1-|z|^2} & |z|u_n^* \end{pmatrix}$

$$\Rightarrow \operatorname{ind}([u_n]) = \left[V_n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} V_n^*\right]$$

$$= \left[\begin{pmatrix} |z|^2 & \sqrt{1-|z|^2}\,|z|u_n \\ \sqrt{1-|z|^2}\,|z|u_n^* & 1-|z|^2 \end{pmatrix}\right]$$

homotopic $\sim \left[\begin{pmatrix} |z|^2 & \sqrt{1-|z|^2}\,|z|u^n \\ \sqrt{1-|z|^2}\,|z|u^{*n} & 1-|z|^2 \end{pmatrix}\right]$

$\Leftrightarrow \quad \tilde{u}_n \sim u^n$

$\Leftrightarrow \quad \operatorname{wind}(\tilde{u}_n) = n.$

Index map in the Toeplitz algebra

$$K_1(\mathcal{T}) \longrightarrow K_1(C(S^1)) \xrightarrow{\text{ind}} K_0(K(\ell_2(\mathbb{N})))$$

Let: $[u_n] \in K_1(C(S^1))$

Lift: $|z| u_n \in C(\bar{D})$

Toeplitz operator: $T_n = P\,|z|\,u_n P$,

$P : L_2(\bar{D}) \longrightarrow A^2(\bar{D}) = L_2(\bar{D}) \cap \mathrm{Hol}($

Bergmann projection

Unitary 2x2-matrix: $V_n := \begin{pmatrix} T_n & -\sqrt{1-T_nT_n^*} \\ \sqrt{1-T_n^*T_n} & T_n^* \end{pmatrix}$

$$\Rightarrow \operatorname{ind}([u_n]) = V_n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} V_n^*$$

$$= \begin{pmatrix} T_nT_n^* & T_n\sqrt{1-T_n^*T_n} \\ T_n^*\sqrt{1-T_nT_n^*} & 1-T_n^*T_n \end{pmatrix}$$

Winding number: $u_n \sim U^n$ (homotopic)

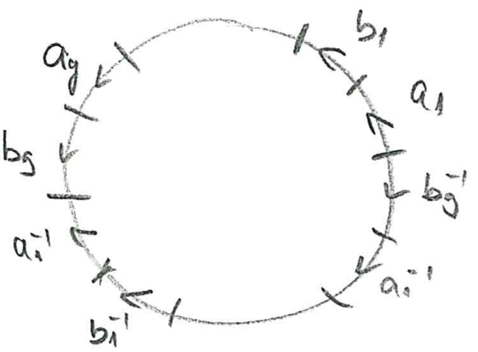
$$\Leftrightarrow \operatorname{wind}(u_n) = n \Leftrightarrow T_n \sim S^n|z|$$

$$\Leftrightarrow \operatorname{Ind}([u_n]) \sim \begin{pmatrix} 1-q^{2n}-q^{2n}|z|^2 & S^n|z|\sqrt{1-|z|^2} \\ |z|\sqrt{1-|z|^2}\,S^{*n} & 1-|z|^2 \end{pmatrix}$$

(Bott-Projections)

The K-Theory of quantum surfaces

Orientable:



C*-algebra extension:

$$0 \longrightarrow C_0(\mathbb{D}) \longrightarrow C(\Pi_g) \longrightarrow C(S^1 \vee \ldots \vee S^1) \longrightarrow \mathbb{C}$$

$S^1 \vee \ldots \vee S^1 =$  $=$ bouquet of g cicles

Quantum:

$$0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \longrightarrow C(\Pi_{g,q}) \longrightarrow C(S^1 \vee \ldots \vee S^1) \longrightarrow \mathbb{C}$$

C*-algebra extension for $S^1 \vee \ldots \vee S^1$

$$0 \longrightarrow C_0((0,1)) \oplus \ldots \oplus C_0((0,1)) \longrightarrow C(S^1 \vee \ldots \vee S^1) \longrightarrow \mathbb{C} \longrightarrow$$

$$\Longrightarrow K_0(C(S^1 \vee \ldots \vee S^1)) = \mathbb{Z} = K_0(\mathbb{C})$$

$$K_1(C(S^1 \vee \ldots \vee S^1)) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$$

$$= K_1(C(S^1)) \oplus \ldots \oplus K_1(C(\ldots$$

6-term exact sequence

$$\begin{array}{ccccc}
K_0(C_0(\mathbb{D})) & \longrightarrow & K_0(C(\Pi_g)) & \longrightarrow & K_0(C(S^1 \vee \dots \vee S^1)) \\
\text{ind} \uparrow & & & & \downarrow \\
K_1(C(S^1 \vee \dots \vee S^1)) & \longleftarrow & K_1(C(\Pi_g)) & \longleftarrow & K_1(C_0(\mathbb{D}))
\end{array}$$

Index map:

Take $[v] \in K_1(C(S^1 \vee \dots \vee S^1))$

Consider $\tilde{v} \in C(S^1) = C(\partial \overline{\mathbb{D}})$

$$\Rightarrow \text{ind}([v]) = \left[\begin{pmatrix} |z|^2 & \sqrt{1-|z|^2}\,|z|\,v \\ \sqrt{1-|z|^2}\,|z|\,v^* & 1-|z|^2 \end{pmatrix} \right]$$

$$= \left[\begin{pmatrix} |z|^2 & \sqrt{1-|z|^2}\,|z|\,u^k \\ \sqrt{1-|z|^2}\,|z|\,u^{*k} & 1-|z|^2 \end{pmatrix} \right]$$

$$\Leftrightarrow \text{wind}(\tilde{v}) = \text{wind}(u^k) = k$$

But: $\text{wind}(\tilde{v}\restriction_{a_j}) = -\text{wind}(\tilde{v}\restriction_{a_j^{-1}})$

$$\Rightarrow \text{wind}(\tilde{v}) = 0$$

$$\Rightarrow \text{ind}([v]) = 0.$$

K-groups:

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r, hook] & K_0(C(\Pi_g)) \arrow[r, two heads] & \mathbb{Z} \arrow[d, "0"] \\
\mathbb{Z}^{2g} \arrow[u, "0"] & K_1(C(\Pi_g)) \arrow[l, "\cong"'] & 0 \arrow[l, "0"]
\end{tikzcd}
$$

$$[1] \longmapsto [1]$$

$$\Longrightarrow \quad K_0(C(\Pi_g)) \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$K_1(C(\Pi_g)) \cong \mathbb{Z}^{2g}$$

Quantum:

$$\begin{array}{ccccc} K_0(K(\ell_2(\mathbb{N}))) & \longrightarrow & K_0(C(\Pi_{g,q})) & \longrightarrow & K_0(C(S^1 \vee \dots \vee S^1)) \\ \uparrow \text{ind} & & & & \downarrow \\ K_1(C(S^1 \vee \dots \vee S^1)) & \longleftarrow & K_1(C(\Pi_{g,q})) & \longleftarrow & K_1(K(\ell_2(\mathbb{N}))) \end{array}$$

Index map

Take $[v] \in K_1(C(S^1 \vee \dots \vee S^1))$

Consider $\tilde{v} \in C(S^1) = C(\partial D)$

Lift: $T = P|z|\tilde{v}P$ – Toeplitz operator

$$\Rightarrow \mathrm{ind}([v]) = \left[\begin{pmatrix} TT^* & T\sqrt{1-T^*T} \\ T^*\sqrt{1-TT^*} & 1-T^*T \end{pmatrix}\right] - \dots$$

$$= \left[\begin{pmatrix} 1-q^{2n}-q^{2n}|z|^2 & S^k|z|\sqrt{1-|z|^2} \\ |z|\sqrt{1-|z|^2}\,S^{*k} & 1-|z|^2 \end{pmatrix}\right] - \dots$$

$\Leftrightarrow \mathrm{wind}(\tilde{v}) = k$

As before: $\mathrm{wind}(\tilde{v}) = 0$

$\Rightarrow \mathrm{ind}([v]) = 0$

K-groups

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, hook] & K_0(C(\mathbb{T}_{g,q})) \arrow[r, two heads] & \mathbb{Z} \arrow[d, "0"] \\
\mathbb{Z}^{2g} \arrow[u, "0"] & K_1(C(\mathbb{T}_{g,q})) \arrow[l, "\cong"'] & 0 \arrow[l, "0"]
\end{tikzcd}$$

$$\Longrightarrow \boxed{\begin{aligned} K_0(C(\mathbb{T}_{g,q})) &\cong \mathbb{Z} \oplus \mathbb{Z} \\ K_1(C(\mathbb{T}_{g,q})) &\cong \mathbb{Z}^{2g} \end{aligned}}$$

Non-orientable

As before, consider

$$\mathrm{ind} : K_1(C(S^1 \vee \dots \vee S^1)) \longrightarrow K_0(C_0(D))$$

Let $[v] \in K_1(C(S^1 \vee \dots \vee S^1))$

Lift $\tilde{v} \in C(S^1) = C(\partial D)$

$$\mathrm{ind}([v]) = \left[\begin{pmatrix} |z|^2 & \sqrt{1-|z|^2}\,|z|\,u^k \\ \sqrt{1-|z|^2}\,|z|\,u^k & 1-|z|^2 \end{pmatrix} \right]$$

$$\Longleftrightarrow \mathrm{wind}(\tilde{v}) = \mathrm{wind}(u^k) = k,$$

Suppose that $\mathrm{wind}(v) = n$

Each circle appears twice in the lift $\Longrightarrow \mathrm{wind}(\tilde{v}) = 2n$

$$\Longrightarrow k = 2n = \sum_{j=1}^{n} 2\,\mathrm{wind}(\tilde{v}\restriction_{a_j})$$

$$= \sum_{j=1}^{n} 2\,\mathrm{wind}(v\restriction_{a_j})$$

$$\Longrightarrow \mathrm{ind}([(n_1, \dots, n_g)]) = \sum_{j=1}^{n} 2n_j$$

K-groups

$$
\begin{tikzcd}
2k_1+\dots+2k_n & \mathbb{Z} \arrow[r, hook, "\iota"] & K_0(C(\mathbb{P}_n)) \arrow[r] & \mathbb{Z} \arrow[d, "0"] \\
(k_1,\dots,k_n) \arrow[u, mapsto] & \mathbb{Z}^n \arrow[u, "\mathrm{ind}"] & K_1(C(\mathbb{P}_n)) \arrow[l, hook'] & 0 \arrow[l, "0"]
\end{tikzcd}
$$

$$[1] \longmapsto [1]$$

$$\mathrm{Im}(\mathrm{ind}) = 2\mathbb{Z} \subset \mathbb{Z}$$

$$\Rightarrow \quad K_0(C(\mathbb{P}_n)) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$$

$$K_1(C(\mathbb{P}_n)) = \mathbb{Z}^{n-1}$$

Quantum:

$$\begin{array}{ccccc}
K_0(\mathcal{K}(\ell^2(\mathbb{N}))) & \longrightarrow & K_0(C(\mathbb{P}_{n,q})) & \longrightarrow & K_0(C(S^1 \vee \dots \vee S^1)) \\
\text{ind} \uparrow & & & & \downarrow \\
K_1(C(S^1 \vee \dots \vee S^1)) & \longleftarrow & K_1(C(\mathbb{P}_{n,q})) & \longleftarrow & K_1(\mathcal{K}(\ell^2(\mathbb{N})))
\end{array}$$

Index map: $[v] \in K_1(C(S^1 \vee \dots \vee S^1))$

Lifts to $\tilde{v} \in C(S^1) = C(\partial \bar{D})$

Each circle $a_j \in C(S^1 \vee \dots \vee S^1)$

appears twice on $\tilde{v}$.

$$\Rightarrow \operatorname{wind}(\tilde{v}) = \sum_{j=1}^{n} 2 \operatorname{wind}(v|_{a_j})$$

$$\Rightarrow \operatorname{ind}([(k_1, \dots, k_n)]) = 2k_1 + \dots + 2k_n.$$

K-groups:

$$\begin{tikzcd}
2k_1+\dots+2k_n & \mathbb{Z} \arrow[r, "\iota"] & K_0(C(\mathbb{P}_{n,q})) \arrow[r] & \mathbb{Z} \arrow[d, "0"] \\
(k_1,\dots,k_n) \arrow[u, mapsto] & \mathbb{Z}^n \arrow[u, "\mathrm{ind}"] & K_1(C(\mathbb{P}_{n,q})) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

$$[1] \longmapsto [1]$$

$$\mathrm{Im}(\mathrm{ind}) = 2\mathbb{Z} \subset \mathbb{Z}$$

$$\Rightarrow \quad K_0(C(\mathbb{P}_{n,q})) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z},$$

$$K_1(C(\mathbb{P}_{n,q})) = \mathbb{Z}^{n-1}.$$