Decidability questions for Cuntz-Krieger algebras and their underlying dynamics

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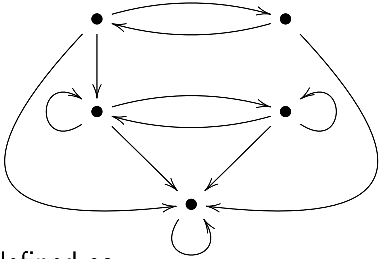
Content

Outline

To a finite graph $E = (E_0, E_1, r, s)$ such as



we associate X_E defined as

$$\mathsf{X}_E = \{(e_n) \in (E_0)^{\mathbb{Z}} \mid r(e_n) = s(e_{n+1})\}$$

Note that X_E is closed in the topology of $(E_0)^{\mathbb{Z}}$ and comes equipped with a shift map $\sigma : \mathsf{X}_E \to \mathsf{X}_E$ which is a homeomorphism. We call X_E a **shift space** (of finite type) over the **alphabet** E_0.

Definition

The **suspension flow** SX of a shift space X is $X \times \mathbb{R}/\sim$ with

$$(x,t) \sim (\sigma(x), t-1)$$

Note that SX has a canonical $\mathbb{R}$-action.

Definitions

Let X and Y be shift spaces.

- X is conjugate to Y (written $X \simeq Y$) if there is a shift-invariant homeomorphism $\varphi : X \to Y$.
- X is flow equivalent to Y (written $X \sim_{\mathrm{FE}} Y$) if there is an orientation-preserving homeomorphism $\psi : SX \to SY$

Question

Are these notions decidable for shifts of finite type?

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Theorem (Boyle-Steinberg)

Flow equivalence is decidable among shifts of finite type.

Definition

Let $A \in M_n(\mathbb{Z}_+)$ and $B \in M_m(\mathbb{Z}_+)$ be given. We say that A is **elementary equivalent** to B if there exist $D \in M_{n\times m}(\mathbb{Z}_+)$ and $E \in M_{m\times n}(\mathbb{Z}_+)$ so that

$$A = DE \qquad B = ED.$$

The smallest equivalence relation on $\bigcup_{n\geq 1} M_n(\mathbb{Z}_+)$ is called **strong shift equivalence**.

Let G_A be the graph with adjacency matrix A. We abbreviate $\mathsf{X}_A = \mathsf{X}_{G_A}$.

Theorem (Williams)

$\mathsf{X}_A \simeq \mathsf{X}_B$ *if and only if* A *is strong shift equivalent to* B.

Definition

We say that that A and B are **shift equivalent** of lag ℓ when there exist $D \in M_{n\times m}(\mathbb{Z}_+)$ and $E \in M_{m\times n}(\mathbb{Z}_+)$ so that

$$A^\ell = DE \qquad B^\ell = ED \qquad AD = DB \qquad EA = BE.$$

Strong shift equivalence implies shift equivalence.

Theorem (Kim-Roush)

Shift equivalence is decidable.

It took decades to disprove

William's conjecture

Shift equivalence coincides with strong shift equivalence.

and indeed it is a prominent open question if conjugacy is decidable for shifts of finite type.

Outline

Singular and regular vertices

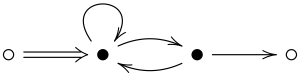
Definitions

Let E be a graph and $v \in E^0$.

- v is a *sink* if $|s^{-1}(\{v\})| = 0$
- v is an *infinite emitter* if $|s^{-1}(\{v\})| = \infty$

Definition

v is *singular* if v is a sink or an infinite emitter. v is *regular* if it is not singular.



Graph algebras

Definition

The *graph C^*-algebra* $C^*(E)$ is given as the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges subject to the Cuntz-Krieger relations

1. $s_e^* s_e = p_{r(e)}$
2. $s_e s_e^* \leq p_{s(e)}$
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every regular v

$C^*(E)$ is unital precisely when E has finitely many vertices.

Observation

$$\gamma_z(p_v) = p_v \qquad \gamma_z(s_e) = zs_e$$

induces a **gauge action** $\mathbb{T} \mapsto \mathrm{Aut}(C^*(E))$

Definition

$$\mathfrak{D}_E = \overline{\mathrm{span}}\{s_\alpha s_\alpha^* \mid \alpha \text{ path of } E\}$$

Note that $\mathfrak{D}_E$ is commutative and that

$$\mathfrak{D}_E \subseteq \mathfrak{F}_E = \{a \in C^*(E) \mid \forall z \in \mathbb{T} : \gamma_z(a) = a\}$$

$\mathfrak{D}_E$ has spectrum X_A when $E = E_A$ arises from an essential and finite matrix A. This fundamental case was studied by Cuntz and Krieger, using the notation $\mathcal{O}_A = C^*(E_A)$.

Theorem (E-Restorff-Ruiz-Sørensen)

$*$-isomorphism and stable $*$-isomorphism of unital graph C^*-algebras is decidable.

Theorem (Carlsen-E-Ortega-Restorff, Matsumoto-Matui)

$$(C^*(E_A) \otimes \mathbb{K}, \mathfrak{D} \otimes c_0) \simeq (C^*(E_B) \otimes \mathbb{K}, \mathfrak{D} \otimes c_0) \iff \mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_B$$

Theorem (Carlsen-Rout, Matsumoto)

$$(C^*(E_A) \otimes \mathbb{K}, \mathfrak{D} \otimes c_0, \gamma \otimes \mathrm{Id}) \simeq (C^*(E_B) \otimes \mathbb{K}, \mathfrak{D} \otimes c_0, \gamma \otimes \mathrm{Id})$$

$$\iff$$

$$\mathsf{X}_A \simeq \mathsf{X}_B$$

Theorem (E-Restorff-Ruiz-Sørensen)

$*$-isomorphism and stable $*$-isomorphism of Cuntz-Krieger algebras is decidable.

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$$\iff$$

$$\mathsf{X}_A \simeq \mathsf{X}_B$$

Outline

Definition

With x, y, z $\in \{0,1\}$ we write

$$E \overset{\text{xyz}}{\longrightarrow} F$$

when there exists a $*$-isomorphism $\varphi : C^*(E) \otimes \mathbb{K} \to C^*(F) \otimes \mathbb{K}$ with additionally satisfies

- $\varphi(1_{C^*(E)} \otimes e_{11}) = 1_{C^*(F)} \otimes e_{11}$ when x $= 1$
- $\varphi \circ (\gamma \otimes \mathrm{Id}) = (\gamma \otimes \mathrm{Id}) \circ \varphi$ when y $= 1$
- $\varphi(\mathfrak{D}_E \otimes c_0) = \mathfrak{D}_F \otimes c_0$ when z $= 1$.

Theorem (E-Restorff-Ruiz-Sørensen)

$E \xrightarrow{\text{x0z}} F$ *is decidable.*

Theorem (Carlsen-E-Ortega-Restorff, Matsumoto-Matui)

$E_A \xrightarrow{001} E_B \iff \mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_B$

Theorem (Carlsen-Rout, Matsumoto)

$E_A \xrightarrow{011} E_B \iff \mathsf{X}_A \simeq \mathsf{X}_B$

Outline

Moves

Move (S)

Remove a regular source, as

Move (R)

Reduce a configuration with a transitional regular vertex, as

or

Moves

Move **(S)**

Remove a regular source, as

Move **(R)**

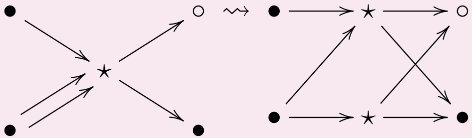
Reduce a configuration with a transitional regular vertex, as

or

Moves

Move (I)

Insplit at regular vertex



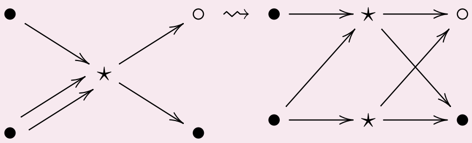
Move (O)

Outsplit at any vertex (at most one group of edges infinite)

Moves

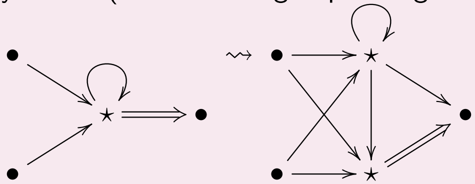
Move (I)

Insplit at regular vertex



Move (O)

Outsplit at any vertex (at most one group of edges infinite)

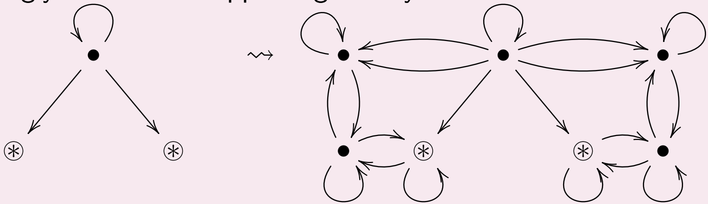


Move **(C)**

"Cuntz splice" on a vertex supporting two cycles

Move **(P)**

"Butterfly move" on a vertex supporting a single cycle emitting only singly to vertices supporting two cycles



Theorem (E-Restorff-Ruiz-Sørensen)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras. Then the following are equivalent

(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) *There is a finite sequence of moves of type*

(S),**(R)**,**(O)**,**(I)**,**(C)**,**(P)**

and their inverses, leading from E to F.