The classification problem for Cuntz-Krieger algebras

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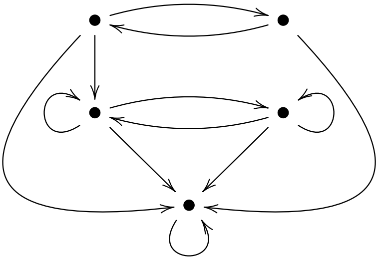
Content

Outline

Throughout we let $A \in M_n(\mathbb{N}_0)$ be **essential**: no zero rows, no zero columns. We consider A as the adjacency matrix of a graph G_A.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Cuntz-Krieger algebras

Definition

$\mathcal{O}_A$ is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in V(G_A)\}$ and partial isometries $\{s_e : e \in E(G_A)\}$ with mutually orthogonal ranges, subject to

1. $s_e^* s_e = p_{r(e)}$
2. $p_v = \sum_{s(e)=v} s_e s_e^*$

Key observations

- $K_0(\mathcal{O}_A) = \operatorname{coker}(A^t - I)$ and $K_1(\mathcal{O}_A) = \ker(A^t - I)$
- $S_i \mapsto \lambda S_i$ induces a gauge action $\mathbb{T} \mapsto \operatorname{Aut}(\mathcal{O}_A)$
- $\mathcal{O}_{[1]} = C(\mathbb{T})$, $\mathcal{O}_{\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]} = M_2(C(\mathbb{T}))$ etc.

Geometric classification problem

Describe the equivalence relation on graphs defined by

$$G_A \sim_{C^*} G_B \Longleftrightarrow \mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K}$$

Symbolic dynamics

Definition

Let $\mathfrak{a}$ be a finite set. A **shift space** over $\mathfrak{a}$ is a subset of $\mathfrak{a}^{\mathbb{Z}}$ which is closed (product topology) and shift invariant.

Edge shift

$$X_A = \{(e_n) \in E(G_A) \mid r(e_n) = s(e_{n+1})\}$$

Definition

The **suspension flow** SX of a shift space X is $X \times \mathbb{R}/\sim$ with

$$(x, t) \sim (\sigma(x), t - 1)$$

Note that SX has a canonical $\mathbb{R}$-action.

Definitions

Let X and Y be shift spaces.

- X is conjugate to Y (written $X \simeq Y$) if there is a shift-invariant homeomorphism $\varphi : X \to Y$.
- X is flow equivalent to Y (written $X \sim_{\mathrm{FE}} Y$) if there is an orientation-preserving homeomorphism $\psi : SX \to SY$

Theorem (Cuntz/Krieger)

$X_A \sim_{\mathrm{FE}} X_B \Longrightarrow \mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K}$

Moves

Move (I)

Insplit at any vertex, e.g.

Move (O)

Outsplit at any vertex, e.g.

Theorem (Williams)

The following are equivalent

1. $X_A \simeq X_B$
2. G_A *can be obtained from* G_B *by a finite number of moves of type* **(I)** *and* **(O)**, *and their inverses.*

Move **(R)**

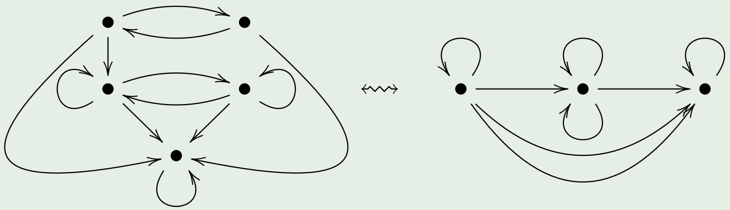
Reduce a configuration with a transitional vertex, e.g.

Theorem (Parry/Sullivan)

The following are equivalent

1. $X_A \sim_{\mathrm{FE}} X_B$
2. *G_A can be obtained from G_B by a finite number of moves of type* **(I)**, **(O)**, *and* **(R)**, *and their inverses.*

Example



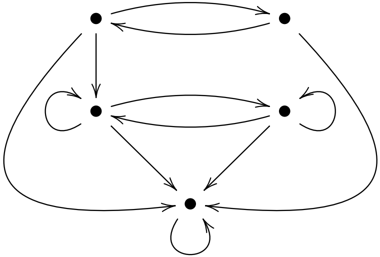
Conclusion

When G_A can be obtained from G_B by a finite number of moves of type **(I)**, **(O)**, and **(R)**, and their inverses, we have that $G_A \sim_{C^*} G_B$.

Outline

The gauge invariant ideals of $\mathcal{O}_A$ can be determined directly from a block structure of A with irreducible diagonal blocks.

$$\left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$



Cuntz' condition (II)

The following are equivalent

1. No irreducible block of A is a permutation matrix
2. All ideals of $\mathcal{O}_A$ are gauge invariant
3. $\mathcal{O}_A$ has a finite number of ideals

Theorem (Rørdam 1995)

The K_0-group $K_0(\mathcal{O}_A)$ is a complete invariant for stable isomorphism of simple Cuntz-Krieger algebras.

Theorem (Rørdam 1997)

The six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathfrak{I}) & \longrightarrow & K_0(\mathcal{O}_A) & \longrightarrow & K_0(\mathcal{O}_A/\mathfrak{I}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_A/\mathfrak{I}) & \longleftarrow & K_1(\mathcal{O}_A) & \longleftarrow & K_1(\mathfrak{I}) \end{array}$$

is a complete invariant for stable isomorphism of Cuntz-Krieger algebras with a unique non-trivial ideal $\mathfrak{I}$.

Theorem (Restorff 2006)

The reduced filtered K-theory $\mathbf{FK}(-)$ consisting of certain partial strands of six-term exact sequences

$$K_1(\mathfrak{J}/\mathfrak{I}) \longrightarrow K_0(\mathfrak{K}/\mathfrak{J}) \longrightarrow K_0(\mathfrak{K}/\mathfrak{I}) \longrightarrow K_0(\mathfrak{J}/\mathfrak{I})$$

arising from a selection of ideals

$$\mathfrak{I} \triangleleft \mathfrak{J} \triangleleft \mathfrak{K} \triangleleft \mathcal{O}_A$$

is a complete invariant for stable isomorphism of Cuntz-Krieger algebras with condition (II).

Example

With

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

we have

$$K_0(\mathcal{O}_A) = \mathbb{Z} = K_0(\mathcal{O}_{[1]})$$

but

$$K_0(\mathcal{O}_A)_+ = \mathbb{Z} \neq \mathbb{N}_0 = K_0(\mathcal{O}_{[1]})_+$$

Observation

The **ordered** K_0-group $K_0(\mathcal{O}_A)$ is a complete invariant for stable isomorphism of **gauge** simple Cuntz-Krieger algebras.

Theorem (E-Restorff-Ruiz-Sørensen)

The **ordered** *reduced filtered K-theory $\mathbf{FK}^{+,\gamma}(-)$ remains a complete invariant for stable isomorphism of Cuntz-Krieger algebras satisfying one of the conditions*

- *The C^*-algebra has at most one non-trivial gauge invariant ideal*
- *The matrix satisfies condition (¬II): Every irreducible component is a permutation matrix.*

The order is trivial in the condition (II) case. The (¬II) case arises naturally in applications such as the quantum lens spaces of Hong/Szymański.

Outline

Theorem (Rørdam 1995)

The following are equivalent when $\mathcal{O}_A$ and $\mathcal{O}_B$ are simple

1. $\mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K}$
2. $K_0(\mathcal{O}_A) \simeq K_0(\mathcal{O}_B)$
3. X_A *is flow equivalent to* X_B *or to* $X_{B'}$ *where*

$$B' = \begin{bmatrix} b_{11} & \cdots & b_{1n} & 0 & 0 \\ \vdots & & \vdots & 0 & 0 \\ b_{n1} & \cdots & b_{nn} & 1 & 0 \\ 0 & \cdots & 1 & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

Cuntz had reduced 3 $\Longrightarrow$ 1 to the case $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ by an argument we will generalize below.

Theorem (Matui-Matsumoto 2013)

The following are equivalent when $\mathcal{O}_A$ and $\mathcal{O}_B$ are simple

1. $(\mathcal{O}_A \otimes \mathbb{K}, \mathcal{D}_A \otimes c_0) \simeq (\mathcal{O}_B \otimes \mathbb{K}, \mathcal{D}_B \otimes c_0)$
2. $K_0(\mathcal{O}_A) \simeq K_0(\mathcal{O}_B)$ *and* $\det(I - A) = \det(I - B)$
3. X_A *is flow equivalent to* X_B
4. G_A *can be obtained from* G_B *by a finite number of moves of type* **(R)**, **(I)** *and* **(O)**, *and their inverses.*

Remark

In recent work by Carlsen/E/Ortega/Restorff we managed to find a different proof of the key component of Matsumoto/Matui which generalizes 1 $\Longleftrightarrow$ 3 to any $\mathcal{O}_A$ of type (II).

Move **(C)**

“Cuntz splice” on a vertex supporting two cycles

Geometric version of Rørdam's theorem

Theorem

The following are equivalent when $\mathcal{O}_A$ and $\mathcal{O}_B$ are simple

1. $\mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K}$
2. $K_0(\mathcal{O}_A) \simeq K_0(\mathcal{O}_B)$
3. *G_A can be obtained from G_B by a finite number of moves of type* **(R)**, **(I)**, **(O)** *and* **(C)**, *and their inverses.*

Theorem (Restorff, E/Ruiz/Sørensen)

When A and B satisfy one of

- *A and B have property (II)*
- *$\mathcal{O}_A$ and $\mathcal{O}_B$ each have at most one non-trivial gauge invariant ideal*
- *A and B have property ($\neg$II)*

the following are equivalent

1. $\mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K}$
2. $\mathbf{FK}^{+,\gamma}(\mathcal{O}_A) \simeq \mathbf{FK}^{+,\gamma}(\mathcal{O}_B)$
3. *G_A can be obtained from G_B by a finite number of moves of type* **(R)**, **(I)**, **(O)** *and* **(C)**, *and their inverses.*

Example

The pair of matrices

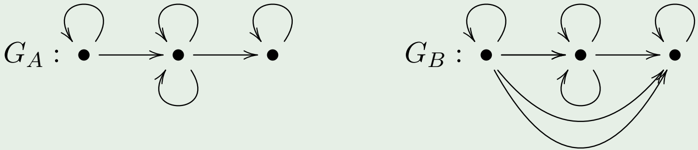
$$\begin{bmatrix} 0 & 1 & 0 & 0 & \mathbf{1} \\ 1 & 0 & 1 & 0 & \mathbf{1} \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & \mathbf{0} \\ 1 & 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is not covered by the previous result.

Outline

Example (E-Restorff-Ruiz-Sørensen)

When



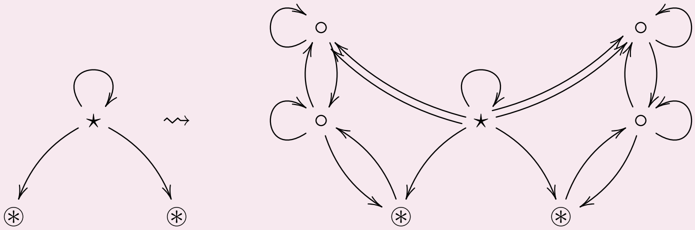
we get that

$$\mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K},$$

yet G_A can not be obtained from G_B by a finite number of moves of type **(R)**, **(I)**, **(O)** and **(C)**, or their inverses.

Move **(P)**

"Pulelehua move" on a vertex supporting a single cycle emitting only singly to vertices supporting two cycles



Theorem (E/Restorff/Ruiz/Sørensen)

The following are equivalent

1. $\mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K}$
2. $\mathbf{FK}^{+,\gamma}(\mathcal{O}_A) \simeq \mathbf{FK}^{+,\gamma}(\mathcal{O}_B)$
3. G_A *can be obtained from* G_B *by a finite number of moves of type* **(R)**, **(I)**, **(O)**, **(C)** *and* **(P)**, *and their inverses.*

- Isomorphism is decidable due to of recent results by Boyle/Steinberg.
- The geometric classification results extend to all unital graph algebras.
- The classification is strong and thus leads to exact classifcation by taking the class of the unit into account.
- By entirely different methods, a complete classification for purely infinite graph algebras with finitely many ideals has been obtained by Bentmann/Meyer.
- No list of moves is known to generate stable isomorphism even for simple (nonunital!) graph algebras.
- There exist two graph algebras each with two non-trivial ideals for which the classification problem is open.