The complete classification of unital graph C^*-algebras

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Content

Outline

Cuntz-Krieger 1980

4. Flow Equivalence

Topological Markov chains are said to be flow equivalent if their suspension flows act on spaces that are homeomorphic under homeomorphisms that respect the orientation of the orbits [11]. Equivalently they are flow equivalent if they induce isomorphic chains on some closed open subset, that is, if they are Kakutani equivalent. Parry and Sullivan have given a description of flow equivalence in terms of a matrix operation [11]. This description leads to a sort of instant computational proof of the invariance of the pair $(\bar{\mathcal{O}}_T, \bar{\mathcal{D}})$ under flow equivalence. We want to give this proof here. We point out, however, that a conceptual proof of this fact is also possible if one exploits the circumstance that $\bar{\mathcal{O}}_T$ arises as a crossed product.

4.1. **Theorem.** *If T_1 and T_2 are flow equivalent then*

$$(\bar{\mathcal{O}}_{T_1}, \bar{\mathcal{D}}) \sim (\bar{\mathcal{O}}_{T_2}, \bar{\mathcal{D}}).$$

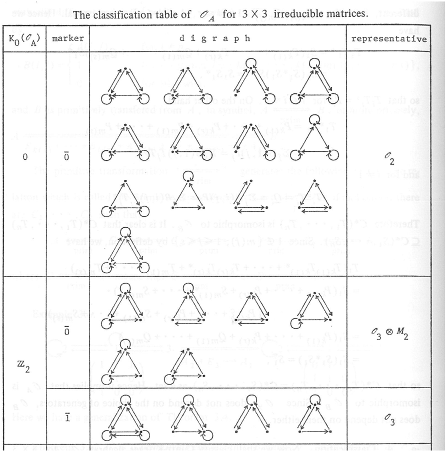
Proof. From the transition matrix $A=(a_{ij})_{1\leq i,j\leq n}$, form the transition matrix

$$\tilde{A}=\begin{pmatrix} 0 & a_{11} & \dots & a_{1n} \\ 1 & 0 & \dots & 0 \\ 0 & a_{21} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

According to Parry and Sullivan, to prove the theorem it is enough to prove that

$$(\bar{\mathcal{O}}_{\bar{\sigma}_A}, \bar{\mathcal{D}}) \sim (\bar{\mathcal{O}}_{\bar{\sigma}_{\tilde{A}}}, \bar{\mathcal{D}}).$$

Enomoto-Fujii-Watatani 1981



The classification table of $\mathcal{O}_A$ for 3×3 irreducible matrices.

$K_0(\mathcal{O}_A)$	marker	digraph	representative
0	$\bar{0}$		$\mathcal{O}_2$
$\mathbb{Z}_2$	$\bar{0}$		$\mathcal{O}_3 \otimes M_2$
	$\bar{1}$		$\mathcal{O}_3$

Rørdam 1995

The class of all simple Cuntz–Krieger algebras is classified by K-theory. This is proved using a theorem of Cuntz, see the appendix, and the two Cuntz–Krieger algebras $\mathcal{O}_2$ and $\mathcal{O}_{2_-}$, where $\mathcal{O}_2$ corresponds to the 1×1 matrix (2) – or the 2×2 matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ – and

$$2_- = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Notice that $\det(1 - 2) = -1$ and $\det(1 - 2_-) = 1$.

LEMMA 6.4. *$\mathcal{O}_2$ is isomorphic to $\mathcal{O}_{2_-}$.*

Proof. Both C^*-algebras have trivial K-theory (both K_0 and K_1 are trivial), and so they are isomorphic by Theorem 6.2. □

The second part of the theorem below is due to Joachim Cuntz.

THEOREM 6.5. *Two simple Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_{A'}$ are stably isomorphic if and only if $K_0(\mathcal{O}_A)$ is isomorphic to $K_0(\mathcal{O}_{A'})$, and $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_{A'}$ if and only if $(K_0(\mathcal{O}_A), [1])$ and $(K_0(\mathcal{O}_{A'}), [1])$ are isomorphic (i.e. if there is a group isomorphism $K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_{A'})$ that carries the class of the unit of $\mathcal{O}_A$ onto the class of the unit of $\mathcal{O}_{A'}$).*

Timeline

Classification results

- 1995: Simple Cuntz-Krieger algebras [Rørdam]
- 1997: Cuntz-Krieger algebras with a unique ideal [Rørdam]
- 2006: Cuntz-Krieger algebras with finitely many ideals [Restorff]
- 2015: All Cuntz-Krieger algebras [E-Restorff-Ruiz-Sørensen]

Outline

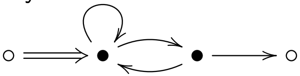
Definition

A graph is a tuple (E^0, E^1, r, s) with

$$r, s : E^1 \to E^0$$

and E^0 and E^1 countable sets.

We think of $e \in E^1$ as an edge from $s(e)$ to $r(e)$ and often represent graphs visually



or by an adjacency matrix

$$\mathsf{A}_E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \infty & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Singular and regular vertices

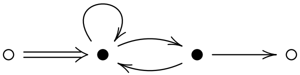
Definitions

Let E be a graph and $v \in E^0$.

- v is a *sink* if $|s^{-1}(\{v\})| = 0$
- v is an *infinite emitter* if $|s^{-1}(\{v\})| = \infty$

Definition

v is *singular* if v is a sink or an infinite emitter. v is *regular* if it is not singular.



Definition

The *graph C^*-algebra* $C^*(E)$ is given as the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges subject to the Cuntz-Krieger relations

1. $s_e^* s_e = p_{r(e)}$
2. $s_e s_e^* \leq p_{s(e)}$
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every regular v

$C^*(E)$ is unital precisely when E has finitely many vertices.

Example

$\mathbb{C}$, $M_2(\mathbb{C})$, $\mathbb{K}$, $\mathcal{O}_2$, $\mathcal{E}_2$, $\mathcal{O}_\infty$, $\mathcal{T}$, $M_{2^\infty} \otimes \mathbb{K}$,$\mathbb{K}^\sim$,...

Observation

$$\gamma_z(p_v) = p_v \qquad \gamma_z(s_e) = z s_e$$

induces a **gauge action** $\mathbb{T} \mapsto \mathrm{Aut}(C^*(E))$

Theorem

Gauge invariant ideals are induced by **hereditary** *and* **saturated** *sets of vertices V:*

- $s(e) \in V \implies r(e) \in V$
- $r(s^{-1}(v)) \subseteq V \implies$ *[$v \in V$ or v is singular]*

and when there are no **breaking vertices**, *all such ideals arise this way.*

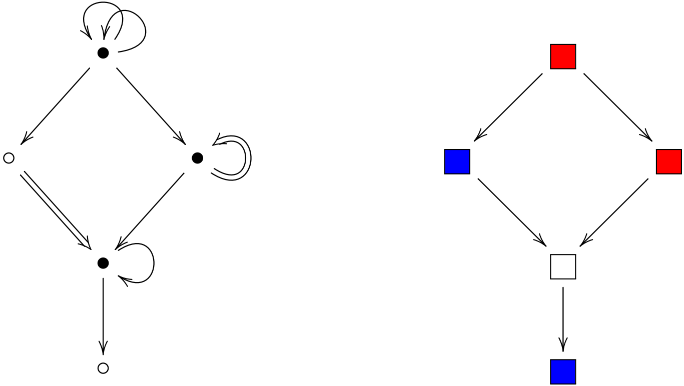
The gauge simple case

Theorem

If a graph C^*-algebra has no non-trivial gauge invariant ideals, it is either

- *a simple AF algebra;*
- *a Kirchberg algebra; or*
- *$C(\mathbb{T}) \otimes \mathbb{K}(H)$ for some Hilbert space H.*

It is easy to tell from the graph which case occurs: The first case occurs when the graph has no cycles; the second when one vertex supports several cycles.



Filtered K-theory

Definition

Let $\mathfrak{A}$ be a C^*-algebra with only finitely many gauge invariant ideals. The collection of all sequences

$$\begin{array}{ccccc} K_0(\mathfrak{J}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{J}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{K}/\mathfrak{J}) & \longleftarrow & K_1(\mathfrak{K}/\mathfrak{I}) & \longleftarrow & K_1(\mathfrak{J}/\mathfrak{I}) \end{array}$$

with gauge invariant $\mathfrak{I} \lhd \mathfrak{J} \lhd \mathfrak{K} \lhd \mathfrak{A}$ is called the *filtered K-theory* of $\mathfrak{A}$ and denoted $\mathsf{FK}^\gamma(\mathfrak{A})$. Equipping all K_0-groups with order we arrive at the *ordered, filtered K-theory* $\mathsf{FK}^{\gamma,+}(\mathfrak{A})$.

Working conjecture [E-Restorff-Ruiz 2010]

$\mathsf{FK}^{\gamma,+}(-)$ is a complete invariant, up to stable isomorphism, for graph C^*-algebras of real rank zero (*i.e.*, with no □ subquotients) and finitely many ideals.

- Confirmed in the non-mixed cases: ■ by Elliott 1976 and ■ by Bentmann-Meyer 2014 amended by Restorff-Ruiz.
- Confirmed by E-Restorff-Ruiz in further cases with controlled mixing, including the case with a single ideal. First open cases:

- No counterexamples are known, even allowing for □ subquotients.

Outline

Move **(S)**

Remove a regular source, as

Move **(R)**

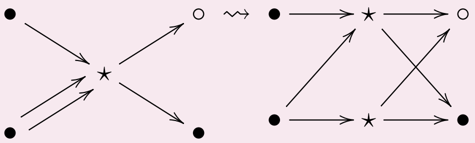
Reduce a configuration with a transitional regular vertex, as

or

Moves

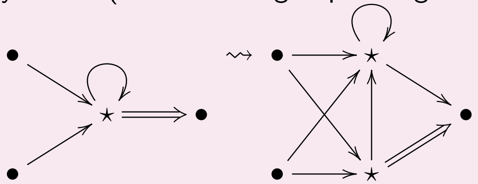
Move **(I)**

Insplit at regular vertex



Move **(O)**

Outsplit at any vertex (at most one group of edges infinite)



Definition

$E \sim_{ME} F$ when there is a finite sequence of moves of type

(S),**(R)**,**(O)**,**(I)**,

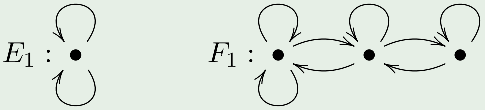
and their inverses, leading from E to F.

Theorem (Cuntz-Krieger, Bates-Pask)

$$E \sim_{ME} F \Longrightarrow C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$$

Example (Rørdam 1995)

When



we get that

$$C^*(E_1) \otimes \mathbb{K} \simeq C^*(F_1) \otimes \mathbb{K},$$

yet $E_1 \not\sim_{ME} F_1$.

Move **(C)**

"Cuntz splice" on a vertex supporting two cycles

Definition

$E \sim_{CE} F$ when there is a finite sequence of moves of type

(S),(R),(O),(I),(C)

and their inverses, leading from E to F.

Theorem (E-Restorff-Ruiz-Sørensen)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras with **real rank zero**. *Then the following are equivalent*

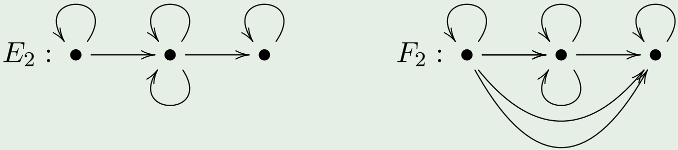
(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) $E \sim_{CE} F$

(iii) $\mathsf{FK}^{\gamma,+}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}(C^*(F))$

Example (E/Restorff/Ruiz/Sørensen 2015)

When



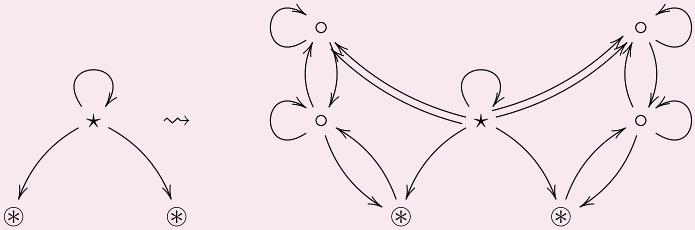
we get that

$$C^*(E_2) \otimes \mathbb{K} \simeq C^*(F_2) \otimes \mathbb{K},$$

yet $E_2 \not\sim_{CE} F_2$.

Move **(P)**

“Butterfly move” on a vertex supporting a single cycle emitting only singly to vertices supporting two cycles



Definition

$E \sim_{PE} F$ when there is a finite sequence of moves of type

(S),(R),(O),(I),(C),(P)

and their inverses, leading from E to F.

Theorem (E-Restorff-Ruiz-Sørensen)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras. Then the following are equivalent

(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) $E \sim_{PE} F$

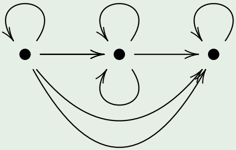
(iii) $\mathsf{FK}^{\gamma,+}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}(C^*(F))$

(iii)$\Longrightarrow$(ii)

Lemma

For any pair of graphs (E, F) *with* $\mathsf{FK}^{\gamma,+}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}(C^*(F))$ *there is a pair of graphs* (E', F') *so that the regular adjacency matrices* $\mathsf{A}^{\circ}_{E'}$ *and* $\mathsf{A}^{\circ}_{F'}$ *have identically, suitably sized upper triangular block matrix forms, and so that* $E \sim_{ME} E'$ *and* $F \sim_{ME} F'$*. We say that* (E', F') *is in* **canonical form***.*

Example

F_2 :  $\rightsquigarrow$

$$\left[\begin{array}{c|ccc|c} 1 & 1 & 0 & 0 & 2 \\ \hline 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(iii)$\Longrightarrow$ (ii)

Proposition

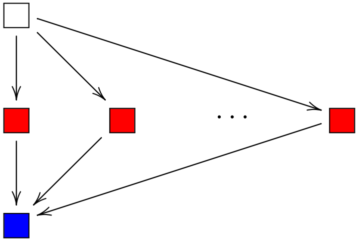
When (E, F) is in canonical form, we have

- $E \sim_{ME} F \iff \exists U, V \in \mathsf{SL}^{\boxplus}(\mathbb{Z}) : U(\mathsf{A}_E - I)^\circ = (\mathsf{A}_F - I)^\circ V$
- $E \sim_{CE} F \iff \exists U, V \in \mathsf{GL}^{\boxplus}(\mathbb{Z}) : U(\mathsf{A}_E - I)^\circ = (\mathsf{A}_F - I)^\circ V$ so that $\det U\{i\} = \det V\{i\} = 1$ at all ■ or □ blocks
- $E \sim_{PE} F \iff \exists U, V \in \mathsf{GL}^{\boxplus}(\mathbb{Z}) : U(\mathsf{A}_E - I)^\circ = (\mathsf{A}_F - I)^\circ V$ so that $\det U\{i\} = 1$ at all ■ or □ blocks

This closely follows an argument in symbolic dynamics by Boyle-Huang. Passing to canonical form is algorithmic. As a consequence (cf. upcoming work by Boyle-Steinberg), stable isomorphism of unital graph C^*-algebras is a decidable property.

(ii)$\Longrightarrow$(i)

General classification methods applied to a very special case of



Outline

Definition

The *Vaksman-Soibelman odd quantum sphere* $C(S_q^{2n-1})$ is the universal C^*-algebra for generators $z_1, \ldots, z_n$ subject to

$$\begin{aligned} z_j z_i &= q z_i z_j \qquad i < j \\ z_j^* z_i &= q z_i z_j^* \qquad i \neq j \\ z_i^* z_i &= z_i z_i^* + (1-q^2) \sum_{j>i} z_j z_j^* \\ 1 &= \sum_{i=1}^{n} z_i z_i^* \end{aligned}$$

for $q \in (0,1)$.

Let n and r be given, set $\theta = e^{2\pi i/r}$ and note that

$$\Lambda_{\underline{m}}(z_i) = \theta^{m_i} z_i$$

with $\underline{m} = (m_1, \ldots, m_n)$ defines $\Lambda_{\underline{m}} \in \operatorname{Aut} C(S_q^{2n-1})$ when $(m_i, r) = 1$ for all i.

Definition [Hong-Szymanski 2002]

Given r, n, and $\underline{m} \in \mathbb{N}^n$. The **quantum lens space** $C(L_q^{2n-1}(r;\underline{m}))$ is the fixed point space

$$C(S_q^{2n-1})^{\Lambda_{\underline{m}}}$$

Theorem (Hong-Szymanski 2002)

$C(L_q^{2n-1}(r;\underline{m}))$ *is a unital graph* C^**-algebra which has real rank one and is postliminal/type I.*

Let us say that $C(L_q^{2n-1}(r;\underline{m}))$ **depends on** $\underline{m}$ when for some $\underline{m}$ and $\underline{m}'$, we have

$$C(L_q^{2n-1}(r;\underline{m})) \not\simeq C(L_q^{2n-1}(r;\underline{m}'))$$

Theorem (E-Restorff-Ruiz-Sørensen, Jensen-Klausen-Rasmussen)

$C(L_q^{2n-1}(r;\underline{m}))$ *depends on* $\underline{m}$ *precisely when*

$$n \geq 2b, \qquad 2b > p > 2, \qquad p \mid r$$

2	3	4	5	6	7	8	9	10	11	12	13
∞	4	6	6	4	8	6	4	6	12	4	14

Outline

Observation [Arklint-Restorff-Ruiz]

$\mathsf{FK}^{\gamma,+}(-)$ fails to give strong classification already for Cuntz-Krieger algebras of real rank zero.

Theorem (E-Restorff-Ruiz-Sørensen)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras. Then the following are equivalent

(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) $E \sim_{PE} F$

(iii) $\mathsf{FK}^{\gamma,+}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}(C^*(F))$

(iv) $\mathsf{FK}^{\gamma,+}_{\mathsf{red}}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}_{\mathsf{red}}(C^*(F))$

Proposition [Carlsen-Restorff-Ruiz]

Any given isomorphism at the level of $\mathsf{FK}^{\gamma,+}_{\mathsf{red}}(-)$ lifts to a pair of $\mathsf{GL}^{\boxplus}(\mathbb{Z})$ matrices (U, V)

Theorem (E-Restorff-Ruiz-Sørensen)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras. Then the following are equivalent

(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) $E \sim_{PE} F$

(iii) $\mathsf{FK}^{\gamma,+}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}(C^*(F))$

(iv) $\mathsf{FK}^{\gamma,+}_{\mathsf{red}}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}_{\mathsf{red}}(C^*(F))$

and any given isomorphism on $\mathsf{FK}^{\gamma,+}_{\mathsf{red}}(-)$ lifts to a $*$-isomorphism.

Corollary [E-Restorff-Ruiz-Sørensen]

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras. Then the following are equivalent

(i) $C^*(E) \simeq C^*(F)$

(ii) $(\mathsf{FK}^{\gamma,+}_{\mathsf{red}}(C^*(E)), [1]) \simeq (\mathsf{FK}^{\gamma,+}_{\mathsf{red}}(C^*(F)), [1])$

$C^*(E)$ contains a canonical abelian subalgebra $\mathcal{D}_E$ which is Cartan under modest assumptions.

Conjecture

The following are equivalent

(i) $E \sim_{ME} F$

(ii) $(C^*(E) \otimes \mathbb{K}, \mathcal{D}_E \otimes c_0) \simeq (C^*(F) \otimes \mathbb{K}, \mathcal{D}_F \otimes c_0)$

Evidence

- (i)$\Longrightarrow$ (ii) holds as noted by Cuntz-Krieger.
- Confirmed when $C^*(E)$ is simple (Matsumoto-Matui 2014, Sørensen 2013)
- Confirmed for Cuntz-Krieger algebras (Arklint-E-Ruiz, Carlsen-E-Restorff-Ruiz)
- $(C^*(E_2) \otimes \mathbb{K}, \mathcal{D}_{E_2} \otimes c_0) \not\simeq (C^*(F_2) \otimes \mathbb{K}, \mathcal{D}_{F_2} \otimes c_0)$