The Modular Gromov-Hausdorff Propinquity

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Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

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4. The *Monge-Kantorovich metric* mk_{L}, defined for any $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ by:

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We call L an *L-seminorm*.

The Dual Gromov-Hausdorff Propinquity

$$(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xleftarrow{\ \pi_{\mathfrak{A}}\ } (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \xrightarrow{\ \pi_{\mathfrak{B}}\ } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$$

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

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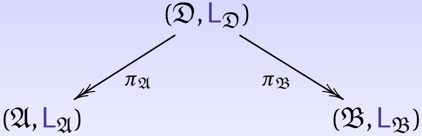

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Definition (The extent of a tunnel)

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{E}))\right) : \mathfrak{E} \in \{\mathfrak{A}, \mathfrak{B}\}\right\}.$$

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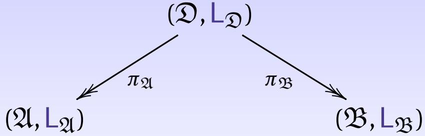

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Definition (L. ,13, 14 / special case)

The *dual propinquity* $\Lambda_F^{*}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

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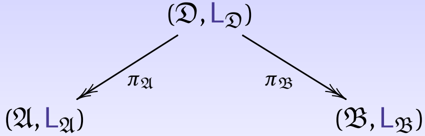

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

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Theorem (L., 13)

The dual propinquity is a *complete metric* up to *full quantum isometry*: $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ iff there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

Example: Gromov-Hausdorff distance (L. 13)

The *dual propinquity* induces the same topology as the *Gromov-Hausdorff distance* on the class of classical compact metric spaces $(C(X), \mathsf{L})$ with (X, d) compact metric space and:

$$\mathsf{L}(f) = \sup\left\{\frac{|f(x) - f(y)|}{\mathsf{d}(x, y)} : x, y \in X, x \neq y\right\}$$

for all $f \in C(X)$ (allowing ∞).

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Example: Quantum Tori (L., 13)

The dual action β of $\mathbb{T}^d$ on any $C^*(\mathbb{Z}^d, \sigma)$ induces an L-seminorm using a continuous length function ℓ via:

$$\forall a \in C^*(\mathbb{Z}^d, \sigma) \quad \mathsf{L}(a) = \sup\left\{ \frac{\|a - \beta^z(a)\|}{\ell(z)} : z \in \mathbb{T} \setminus \{1\} \right\}.$$

Quantum tori form a *continuous family* and can be approximated by *finite dimensional "fuzzy tori"*.

Bridges

Definition (Bridge)

A *bridge* $(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from $\mathfrak{A}$ to $\mathfrak{B}$ is a unital C*-algebra $\mathfrak{D}$, an element $x \in \mathfrak{D}$ and two unital *-monomorphisms $\pi_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{D}$ and $\pi_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{D}$.

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Theorem (L., 13)

Let $(\mathfrak{A}_j, \mathsf{L}_j)$ for $j \in \{1,2\}$ be quasi-Leibniz quantum compact metric spaces and $\gamma = (\mathfrak{D}, x, \pi_1, \pi_2)$ a bridge from $\mathfrak{A}_1$ to $\mathfrak{A}_2$. If λ is the maximum of:

$$\max_{\{j,k\}=\{1,2\}} \sup_{\substack{a_j \in \mathfrak{A}_j \\ \mathsf{L}_j(a_j) \leqslant 1}} \inf_{\substack{a_k \in \mathfrak{A}_k \\ \mathsf{L}_k(a_k) \leqslant 1}} \|\pi_1(a_1)x - x\pi_2(a_2)\|_{\mathfrak{D}}$$

and $\max_{j \in \{1,2\}} \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_j}}\left(\mathscr{S}(\mathfrak{A}_j), \{\varphi \circ \pi_j : \varphi(x \cdot) = \varphi = \varphi(\cdot x)\}\right)$. Then we can build a tunnel on $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ of extend no more than 2λ of the form $\mathsf{L}(a,b) = \max\left\{\mathsf{L}_1(a), \mathsf{L}_2(b), \frac{1}{\lambda}\|\pi_1(a)x - x\pi_2(b)\|_{\mathfrak{D}}\right\}$ for $(a,b) \in \mathfrak{sa}(\mathfrak{A}_1) \oplus \mathfrak{sa}(\mathfrak{A}_2)$ if $\lambda > 0$.

Metrics for Vector Bundles

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- M is a metric space for the path metric m induced by g. Let L be the associated Lipschitz seminorm.
- Let ΓV be the space of continuous sections of V over M, endowed with:

$$\langle \omega, \xi \rangle_{C(M)} : x \in M \longmapsto h_x(\omega_x, \xi_x) \in C(M).$$

Thus, ΓV is a $C(M)$-Hilbert module.

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- Let ∇ be a metric connection on ΓV, i.e.:

$$d_X \langle \omega, \xi \rangle = \langle \nabla_X \omega, \xi \rangle + \langle \omega, \nabla_X \xi \rangle.$$

  ∇ defines a norm on a dense subspace of ΓV:

$$\mathsf{D}(\omega) = \max\left\{ \sqrt{\langle \omega, \omega \rangle_{C(M)}}, |||\nabla \omega|||_{\Gamma V}^{\Gamma T M} \right\}.$$

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Our idea is to introduce a metric on objects of the form $(\Gamma V, \langle \cdot, \cdot \rangle_{C(M)}, \mathsf{D}, C(M), \mathsf{L})$.

Metrized quantum vector bundles

Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is given by:

1. $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space,
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    1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
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Example: Free modules

Given $(\mathfrak{A}, \mathsf{L})$, we set $\langle(a_1,\ldots,a_d),(b_1,\ldots,b_d)\rangle_d = \sum_{j=1}^{d} a_j b_j^*$ and $\mathsf{L}_d(a_1,\ldots,a_d) = \max\left\{\mathsf{L}(\Re a_j), \mathsf{L}(\Im a_j) : j \in \{1,\ldots,d\}\right\}$. Let $\mathsf{D} = \max\{\|\cdot\|_d, \mathsf{L}_d\}$. Then $(\mathfrak{A}^d, \langle,\rangle_d, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is a metrized quantum vector bundle.

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Example: Classical picture

Hermitian bundles over compact connected Riemannian manifolds.

Full quantum isometries

(θ, Θ) full quantum isometry when θ full quantum isometry between bases and $\Theta(a\xi) = \theta(a)\Theta(\xi)$, Θ linear isomorphism

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Example: Heisenberg Modules

Heisenberg modules and their natural *connection*, as build by Connes (81), are (non-free, finitely generated, projective) metrized quantum vector bundles.

The Heisenberg Modules (Connes, 81; Rieffel)

Fix $\theta \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $d \in \mathbb{N} \setminus \{0\}$ such that $\eth = \theta - \frac{p}{q} \neq 0$. Let $\mathscr{A}_\theta$ be the associated quantum torus.

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1. Start with a representation of $\left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R}^3 \right\}$ on $L^2(\mathbb{R})$:

$$\alpha_{\eth}^{x,y,t}\xi(s) = \exp(i\pi(t + 2xs))\xi(s + \eth y).$$

   Promote it to $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.

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2. Let $W_1, W_2 \in U(d)$ with $W_1 W_2 = e^{2i\pi p/q} W_2 W_1$ and $W_1^n = W_2^n = 1$. We get a $\mathscr{A}_\theta = C^*(u_\theta, v_\theta)$-module with:
$$(u_\theta^n v_\theta^m)\xi = W_1^n W_2^m \alpha_\eth^{n,m,0}\xi.$$

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$$(u_\theta^n v_\theta^m)\xi = W_1^n W_2^m \alpha_{\eth}^{n,m,0}\xi.$$
3. For Schwarz functions ξ, ω, set:
$$\langle \xi, \omega \rangle_{\mathscr{H}_\theta^{p,q,d}} = \sum_{n,m \in \mathbb{Z}} \langle u_\theta^n v_\theta^m \xi, \omega \rangle_{L^2(\mathbb{R},\mathbb{C}^d)} u_\theta^n v_\theta^m;$$
complete space of Schwarz functions to the *Heisenberg module* $\mathscr{H}_\theta^{p,q,d}$.

D-norms for Heisenberg Modules

Theorem (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

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Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

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$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

As a note, $\mathsf{D}_\theta^{p,q,d}(\xi)$ is actually the operator norm of $\nabla\xi$ where ∇ is the Connes connection on $\mathscr{H}_\theta^{p,q,d}$.

Bridges for modules

Fix $\Omega_{\mathfrak{A}} = (\mathscr{M}_{\mathfrak{A}}, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $\Omega_{\mathfrak{B}} = (\mathscr{M}_{\mathfrak{B}}, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two metrized quantum vector bundles.

Definition (L., 16)

A *modular bridge* $\left(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j)_{j\in J}, (\eta_j)_{j\in J}\right)$ is a bridge $(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ and two families $(\omega_j)_{j\in J} \in \mathscr{M}_{\mathfrak{A}}$, $(\eta_j)_{j\in J} \in \mathscr{M}_{\mathfrak{B}}$ with $\mathsf{D}_{\mathfrak{A}}(\omega_j), \mathsf{D}_{\mathfrak{B}}(\eta_j) \leqslant 1$ for all $j \in J$.

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Definition (L., 16)

The *length* of a modular bridge is the maximum of the length of its basic bridge, and the sum of:

1. the maximum of $\mathsf{Haus}_k(\{\omega_j : j \in J\}, \{\omega : \mathsf{D}_{\mathfrak{A}}(\omega) \leqslant 1)$ and its counterpart in $\Omega_{\mathfrak{B}}$, where:

$$k(\omega,\xi) = \sup\left\{\|\langle\omega,\eta\rangle_{\mathfrak{A}} - \langle\xi,\eta\rangle_{\mathfrak{A}}\|_{\mathfrak{A}} : \mathsf{D}_{\mathfrak{A}}(\eta) \leqslant 1\right\},$$

2. $\max\left\{\|\pi_{\mathfrak{A}}(\langle\omega_j,\omega_k\rangle_{\mathfrak{A}})x - x\pi_{\mathfrak{B}}(\langle\eta_j,\eta_k\rangle_{\mathfrak{B}})\|_{\mathfrak{D}} : j \in J\right\}$.

The modular propinquity

Definition (L., 16)

The *modular propinquity* is the largest pseudo-metric Λ^{mod} such that $\Lambda^{\mathsf{mod}}(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}) \leqslant \lambda\left(\gamma\right)$ for any modular γ from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$.

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Theorem (Heisenberg Modules (L. 17))

If $(\theta_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}\setminus\mathbb{Q}$ converging to some irrational number θ, and $p,q\in\mathbb{Z}\setminus\{0\}, d\in\mathbb{N}\setminus\{0\}$ then:

$$\lim_{n\to\infty}\Lambda^{\mathsf{mod}}((\mathscr{H}_{\theta_n}^{p,q,d}, \mathscr{H}_{\theta}^{p,q,d})) = 0.$$

Theorem (L., 17)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and p, q, d fixed. If for all $\theta \in \mathbb{R}$, and $a \in \mathscr{A}_\theta$:

$$\mathsf{L}_\theta(a) = \sup\left\{ \frac{\left\|\beta_\theta^{\exp(ix),\exp(iy)} a - a\right\|_{\mathscr{A}_\theta}}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where β_θ is the dual action, and for all $\xi \in \mathscr{H}_\theta^{p,q,d}$ we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where $\eth = \theta - p/q$, then:

$$\lim_{\vartheta\to\theta} \Lambda^{\mathsf{mod}}\left(\left(\mathscr{H}_\vartheta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\vartheta^{p,q,d}}, \mathsf{D}_\vartheta^{p,q,d}, \mathscr{A}_\vartheta, \mathsf{L}_\vartheta\right), \left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)\right) = 0.$$

Thank you!