Orbit and flow equivalence versus diagonal-preserving $*$-isomorphism of Cuntz-Krieger algebras

Søren Eilers
eilers@math.ku.dk

Department of Mathematical Sciences
University of Copenhagen

July 22, 2016

Content

Outline

Main result

Theorem

$$X_A \sim_{\mathrm{FE}} X_B \iff (\mathcal{O}_A \otimes \mathbb{K}, \mathcal{C}_A \otimes c_0) \simeq (\mathcal{O}_B \otimes \mathbb{K}, \mathcal{C}_B \otimes c_0)$$

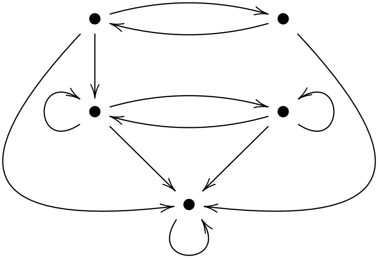
Notes

- 1980: $\Longrightarrow$ observed by Cuntz and Krieger
- 2014: $\Longleftarrow$ proved by Matsumoto and Matui when $\mathcal{O}_A$ and $\mathcal{O}_B$ are simple.
- 2016: $\Longleftarrow$ in general by work with Arklint, Carlsen, Ortega, Restorff, and Ruiz.

Matrices and graphs

Throughout we let $A \in M_n(\mathbb{N}_0)$ be **essential**: no zero rows, no zero columns. We consider A as the adjacency matrix of a graph G_A.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Cuntz-Krieger algebras

Definition

$\mathcal{O}_A$ is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in V(G_A)\}$ and partial isometries $\{s_e : e \in E(G_A)\}$ with mutually orthogonal ranges, subject to

1. $s_e^* s_e = p_{r(e)}$
2. $p_v = \sum_{s(e)=v} s_e s_e^*$

Key observations

- $K_0(\mathcal{O}_A) = \mathrm{coker}(A^t - I)$ and $K_1(\mathcal{O}_A) = \ker(A^t - I)$
- $s_e \mapsto \lambda s_e, p_v \mapsto p_v$ induces a gauge action $\mathbb{T} \mapsto \mathrm{Aut}(\mathcal{O}_A)$

Symbolic dynamics

Let $\mathfrak{a}$ be a finite set and consider $\sigma : \mathfrak{a}^{\mathbb{Z}} \to \mathfrak{a}^{\mathbb{Z}}$ given by

$$\sigma((x_n)) = (x_{n+1})$$

Note that also $\sigma : \mathfrak{a}^{\mathbb{N}} \to \mathfrak{a}^{\mathbb{N}}$ makes sense.

Definition

A **two-sided shift space** over $\mathfrak{a}$ is a subset of $\mathfrak{a}^{\mathbb{Z}}$ which is closed (product topology) and shift invariant. A **one-sided shift space** over $\mathfrak{a}$ is a subset of $\mathfrak{a}^{\mathbb{N}}$ which is closed and shift invariant.

Edge shifts

$$\begin{aligned} X_A &= \{(e_n) \in E(G_A)^{\mathbb{Z}} \mid r(e_n) = s(e_{n+1})\} \\ X_A^+ &= \{(e_n) \in E(G_A)^{\mathbb{N}} \mid r(e_n) = s(e_{n+1})\} \end{aligned}$$

Flow equivalence

Definition

The **suspension flow** SX of a shift space X is $X \times \mathbb{R}/\sim$ with

$$(x,t) \sim (\sigma(x), t-1)$$

Note that SX has a canonical $\mathbb{R}$-action.

Definition

Let X and Y be two-sided shift spaces. X is flow equivalent to Y (written $X \sim_{\mathrm{FE}} Y$) if there is an orientation-preserving homeomorphism $\psi : SX \to SY$.

Definition

A shift space is *irreducible* if some orbit

$$\{\sigma^k(x) \mid k \in \mathbb{Z}\}$$

is dense.

Lemma

The following are equivalent:

1. $\mathcal{O}_A$ is simple
2. X_A is irreducible and infinite (as a set)
3. G_A is strongly connected and not a single cycle

Status 1995

$$
\begin{array}{ccccc}
 & & & & \begin{array}{c}(K_0(\mathcal{O}_A),[1_{\mathcal{O}_A}]) \\ \simeq \\ (K_0(\mathcal{O}_B),[1_{\mathcal{O}_B}])\end{array} \\
 & & & & \Updownarrow \\
X_A^+ \simeq X_B^+ & & \Longrightarrow & & \mathcal{O}_A \simeq \mathcal{O}_B \\
\Downarrow & & & & \Downarrow \\
X_A \simeq X_B & \Longrightarrow & X_A \sim_{\mathrm{FE}} X_B & \Longrightarrow & \mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K} \\
 & & & & \Updownarrow \\
 & & & & K_0(\mathcal{O}_A) \simeq K_0(\mathcal{O}_B)
\end{array}
$$

Is it easy to see that the **diagonal** $\mathcal{C}_A \subseteq \mathcal{O}_A$ given by

$$\mathcal{C}_A = \{s_\mu s_\mu^* \mid \mu = e_1 \cdots e_n\}$$

is abelian, and in fact

Observation [Cuntz/Krieger 1980]

$X_A \sim_{\mathrm{FE}} X_B \Longrightarrow (\mathcal{O}_A \otimes \mathbb{K}, \mathcal{C}_A \otimes c_0) \simeq (\mathcal{O}_B \otimes \mathbb{K}, \mathcal{C}_B \otimes c_0)$

Elaborated status 1995

$$
\begin{array}{ccccc}
 & & \begin{array}{c}(K_0(\mathcal{O}_A),[1_{\mathcal{O}_A}])\\ \simeq \\ (K_0(\mathcal{O}_B),[1_{\mathcal{O}_B}])\end{array} & \Longleftrightarrow & \mathcal{O}_A \simeq \mathcal{O}_B \\
 & & & & \Big\Uparrow \\
X_A^+ \simeq X_B^+ & & \Longrightarrow & & (\mathcal{O}_A,\mathcal{C}_A) \simeq (\mathcal{O}_B,\mathcal{C}_B) \\
\Big\Downarrow & & & & \Big\Downarrow \\
X_A \simeq X_B & \Longrightarrow & X_A \sim_{\mathrm{FE}} X_B & \Longrightarrow & \begin{array}{c}(\mathcal{O}_A\otimes\mathbb{K},\mathcal{C}_A\otimes c_0)\\ \simeq \\ (\mathcal{O}_B\otimes\mathbb{K},\mathcal{C}_B\otimes c_0)\end{array} \\
 & & & & \Big\Downarrow \\
 & & K_0(\mathcal{O}_A) \simeq K_0(\mathcal{O}_B) & \Longleftrightarrow & \mathcal{O}_A\otimes\mathbb{K} \simeq \mathcal{O}_B\otimes\mathbb{K}
\end{array}
$$

Definition

Let X^+ and Y^+ be one-sided shift spaces. A homeomorphism $h : X^+ \to Y^+$ is a *continuous orbit equivalence* if there exist **continuous** maps $k, l : X^+ \to \mathbb{N}_0$ and $k', l' : Y^+ \to \mathbb{N}_0$ such that

$$\sigma_Y^{k(x)}(h(\sigma_X(x))) = \sigma_Y^{l(x)}(h(x))$$

and

$$\sigma_X^{k'(y)}(h^{-1}(\sigma_Y(y))) = \sigma_X^{l'(y)}(h^{-1}(y))$$

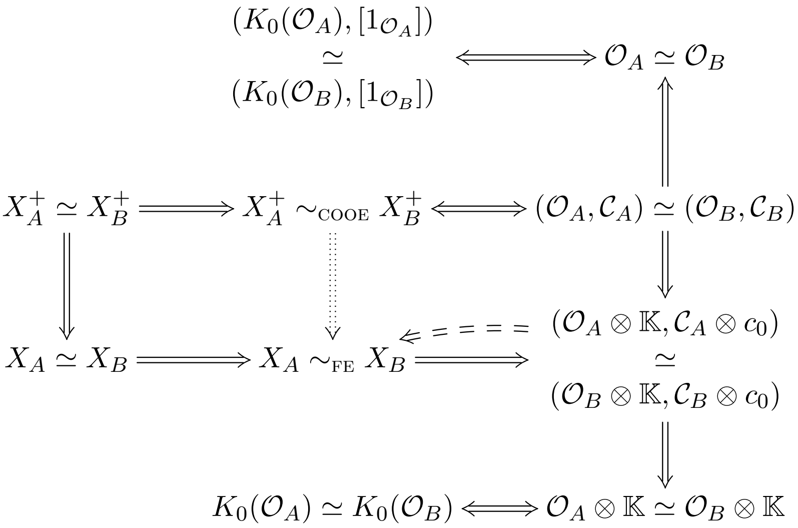
for $x \in X^+$ and $y \in Y^+$. We write $X^+ \sim_{\text{COOE}} Y^+$ in this case.

Theorem (Matsumoto)

When $\mathcal{O}_A$ and $\mathcal{O}_B$ are simple, we have

$$X_A^+ \sim_{\text{COOE}} X_B^+ \iff (\mathcal{O}_A, \mathcal{C}_A) \simeq (\mathcal{O}_B, \mathcal{C}_B)$$

The Matsumoto-Matui approach



Theorem (Matsumoto-Matui)

When $\mathcal{O}_A$ and $\mathcal{O}_B$ are simple, we have

$$X_A^+ \sim_{\text{COOE}} X_B^+ \Longrightarrow X_A \sim_{\text{FE}} X_B$$

The Matsumoto-Matui result

$$
\begin{array}{ccccc}
 & & \begin{array}{c}(K_0(\mathcal{O}_A),[1_{\mathcal{O}_A}])\\ \simeq \\ (K_0(\mathcal{O}_B),[1_{\mathcal{O}_B}])\end{array} & \Longleftrightarrow & \mathcal{O}_A \simeq \mathcal{O}_B \\
 & & & & \Uparrow \\
X_A^+ \simeq X_B^+ & \Longrightarrow & X_A^+ \sim_{\text{COOE}} X_B^+ & \Longleftrightarrow & (\mathcal{O}_A,\mathcal{C}_A) \simeq (\mathcal{O}_B,\mathcal{C}_B) \\
\Downarrow & & \Downarrow & & \Downarrow \\
X_A \simeq X_B & \Longrightarrow & X_A \sim_{\text{FE}} X_B & \Longleftrightarrow & \begin{array}{c}(\mathcal{O}_A\otimes\mathbb{K},\mathcal{C}_A\otimes c_0)\\ \simeq \\ (\mathcal{O}_B\otimes\mathbb{K},\mathcal{C}_B\otimes c_0)\end{array} \\
 & & & & \Downarrow \\
 & & K_0(\mathcal{O}_A) \simeq K_0(\mathcal{O}_B) & \Longleftrightarrow & \mathcal{O}_A\otimes\mathbb{K} \simeq \mathcal{O}_B\otimes\mathbb{K}
\end{array}
$$

Outline

Salvage to general A, B:

$$
\begin{array}{ccccc}
 & & \begin{array}{c}(K_0(\mathcal{O}_A),[1_{\mathcal{O}_A}])\\ \simeq \\ (K_0(\mathcal{O}_B),[1_{\mathcal{O}_B}])\end{array} & \Longleftarrow & \mathcal{O}_A \simeq \mathcal{O}_B \\
 & & & & \Big\Uparrow \\
X_A^+ \simeq X_B^+ & \Longrightarrow & X_A^+ \sim_{\text{COOE}} X_B^+ & \Longleftarrow & (\mathcal{O}_A, \mathcal{C}_A) \simeq (\mathcal{O}_B, \mathcal{C}_B) \\
\Big\Downarrow & & & & \Big\Downarrow \\
X_A \simeq X_B & \Longrightarrow & X_A \sim_{\text{FE}} X_B & \Longrightarrow & \begin{array}{c}(\mathcal{O}_A \otimes \mathbb{K}, \mathcal{C}_A \otimes c_0)\\ \simeq \\ (\mathcal{O}_B \otimes \mathbb{K}, \mathcal{C}_B \otimes c_0)\end{array} \\
 & & & & \Big\Downarrow \\
 & & K_0(\mathcal{O}_A) \simeq K_0(\mathcal{O}_B) & \Longleftarrow & \mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K}
\end{array}
$$

Theorem (E/Restorff/Ruiz/Sørensen)

$$\mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K} \iff \mathsf{FK}_\gamma^+(\mathcal{O}_A) \simeq \mathsf{FK}_\gamma^+(\mathcal{O}_B)$$

and

$$\mathcal{O}_A \simeq \mathcal{O}_B \iff (\mathsf{FK}_\gamma^+(\mathcal{O}_A), [1_{\mathcal{O}_A}]) \simeq (\mathsf{FK}_\gamma^+(\mathcal{O}_B), [1_{\mathcal{O}_B}])$$

$$
\begin{array}{ccccc}
 & & \begin{array}{c}(\mathsf{FK}^+_\gamma(\mathcal{O}_A),[1_{\mathcal{O}_A}])\\ \simeq \\ (\mathsf{FK}^+_\gamma(\mathcal{O}_B),[1_{\mathcal{O}_B}])\end{array} & \Longleftrightarrow & \mathcal{O}_A \simeq \mathcal{O}_B \\
 & & & & \Big\Uparrow \\
X_A^+ \simeq X_B^+ & \Longrightarrow & X_A^+ \sim_{\text{COOE}} X_B^+ & \Longleftarrow & (\mathcal{O}_A,\mathcal{C}_A) \simeq (\mathcal{O}_B,\mathcal{C}_B) \\
\Big\Downarrow & & & & \Big\Downarrow \\
X_A \simeq X_B & \Longrightarrow & X_A \sim_{\text{FE}} X_B & \Longrightarrow & \begin{array}{c}(\mathcal{O}_A\otimes\mathbb{K},\mathcal{C}_A\otimes c_0)\\ \simeq \\ (\mathcal{O}_B\otimes\mathbb{K},\mathcal{C}_B\otimes c_0)\end{array} \\
 & & & & \Big\Downarrow \\
 & & \mathsf{FK}^+_\gamma(\mathcal{O}_A) \simeq \mathsf{FK}^+_\gamma(\mathcal{O}_B) & \Longleftrightarrow & \mathcal{O}_A\otimes\mathbb{K} \simeq \mathcal{O}_B\otimes\mathbb{K}
\end{array}
$$

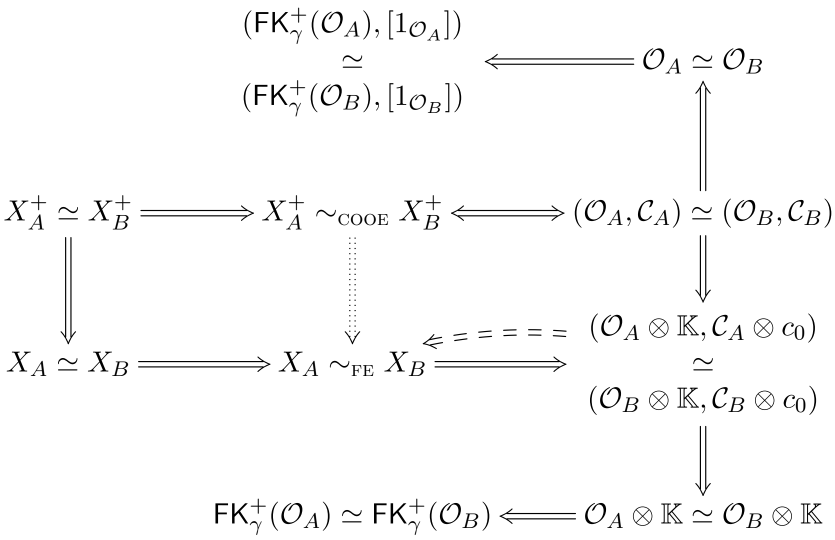
Theorem (Arklint/E/Ruiz, Carlsen/Winger)

$$X_A^+ \sim_{\text{COOE}} X_B^+ \iff (\mathcal{O}_A, \mathcal{C}_A) \simeq (\mathcal{O}_B, \mathcal{C}_B)$$

Notes

When there are no isolated points in X_A^+, this was proved by Brownlowe/Carlsen/Whittaker as a special case of a complete analysis of continuous orbit equivalence for graph C^*-algebras.

The Matsumoto-Matui approach



Outline

The irreducible case

Definition (Ordered cohomology)

For any (one- or twosided) shift space X we let

$$\mathsf{H}_X = \frac{C(X,\mathbb{Z})}{\{f - f\circ\sigma\}}$$

ordered by

$$\mathsf{H}_X^+ = \frac{C(X,\mathbb{Z})^+}{\{f - f\circ\sigma\}}$$

Theorem (Boyle/Handelman)

For irreducible X_A, X_B we have

$$(\mathsf{H}_{X_A}, \mathsf{H}_{X_A}^+) \simeq (\mathsf{H}_{X_B}, \mathsf{H}_{X_B}^+) \iff X_A \sim_{\mathrm{FE}} X_B$$

The irreducible case

Theorem (Boyle/Handelman)

For irreducible X_A, X_B we have

$$(\mathsf{H}_{X_A}, \mathsf{H}^+_{X_A}) \simeq (\mathsf{H}_{X_B}, \mathsf{H}^+_{X_B}) \iff X_A \sim_{\mathrm{FE}} X_B$$

The approach of Matsumoto/Matui is to show

- $X_A^+ \sim_{\mathrm{COOE}} X_B^+ \Longrightarrow (\mathsf{H}_{X_A^+}, \mathsf{H}^+_{X_A^+}) \simeq (\mathsf{H}_{X_B^+}, \mathsf{H}^+_{X_B^+})$
- $(\mathsf{H}_{X_A^+}, \mathsf{H}^+_{X_A^+}) \simeq (\mathsf{H}_{X_A}, \mathsf{H}^+_{X_A})$.

Periodic words

Definition

Let $x \in X^+$. When $\sigma^p(x) = \sigma^q(x)$ for some $p > q$ we say that x is *eventually periodic* and set

$$\mathrm{lp}(x) = \min\{p - q \mid \sigma^p(x) = \sigma^q(x), p > q\}$$

Lemma (Matsumoto/Matui)

When h is given by a continuous orbit equivalence, and x is eventually periodic, then so is $h(x)$.

Definition

Let X^+ and Y^+ be one-sided shift spaces. A homeomorphism $h : X^+ \to Y^+$ is a *continuous orbit equivalence* if there exist continuous maps $k, l : X^+ \to \mathbb{N}_0$ and $k', l' : Y^+ \to \mathbb{N}_0$ such that

$$\sigma_Y^{k(x)}(h(\sigma_X(x))) = \sigma_Y^{l(x)}(h(x))$$

and

$$\sigma_X^{k'(y)}(h^{-1}(\sigma_Y(y))) = \sigma_X^{l'(y)}(h^{-1}(y))$$

for $x \in X^+$ and $y \in Y^+$. We write $X^+ \sim_{\text{COOE}} Y^+$ in this case.

Key proposition (Carlsen/E/Ortega/Restorff)

If a continuous orbit equivalence from X_A^+ to X_B^+ is given by h, k, l, k', l' so that

- $[k - l] \in \mathsf{H}^+_{X_A^+}$
- $[k' - l'] \in \mathsf{H}^+_{X_B^+}$
- $\operatorname{lp}(h(x)) = \sum_{i=0}^{\operatorname{lp}(x)-1}(l(\sigma_X^i(x)) - k(\sigma_X^i(x)))$
- $\operatorname{lp}(h^{-1}(y)) = \sum_{i=0}^{\operatorname{lp}(y)-1}(l'(\sigma_Y^i(y)) - k'(\sigma_Y^i(y)))$

then $X_A \sim_{\mathrm{FE}} X_B$.

The associated groupoid

$$\mathcal{G}_{X^+} = \left\{ (x,n,x') \in X^+ \times \mathbb{Z} \times X^+ \,\middle|\, \exists r,s : \begin{array}{c} n = r - s \\ \sigma_X^r(x) = \sigma_X^s(x') \end{array} \right\}$$

Theorem

The following are equivalent

1. $X_A^+ \sim_{\mathrm{COOE}} X_B^+$
2. $\mathcal{G}_{X_A^+} \simeq \mathcal{G}_{X_B^+}$
3. $(\mathcal{O}_A, \mathcal{C}_A) \simeq (\mathcal{O}_B, \mathcal{C}_B)$

A groupoid isomorphism $\psi : \mathcal{G}_{X_A^+} \to \mathcal{G}_{X_B^+}$ induces an orbit equivalence h by

$$\psi(x,0,x) = (h(x),0,h(x))$$

Theorem (Matsumoto/Matui)

There is a canonical isomorphism $\Phi : H^1(\mathcal{G}_{X_A^+}) \to \mathsf{H}_{X_A^+}$ *having the property that* $\Phi([f]) \in \mathsf{H}_{X_A^+}$ *precisely when*

$$f((x, \mathrm{lp}(x), x)) \geq 0$$

for all eventually periodic x*.*

Observation

When $\psi : \mathcal{G}_{X_A^+} \to \mathcal{G}_{X_B^+}$ is a groupoid isomorphism, we have

$$\psi((x, \mathrm{lp}(x), x)) = (h(x), \pm \mathrm{lp}(h(x)), h(x))$$

for every eventually periodic x, and in fact

$$\psi((x, \mathrm{lp}(x), x)) = (h(x), \mathrm{lp}(h(x)), h(x)) \qquad (\dagger)$$

when x is not an isolated point.

Lemma

When ψ satisfies (†) for every eventually periodic x, then the map $\psi^\flat$ in

$$\begin{array}{ccc} H^1(\mathcal{G}_{X_A^+}) & \longrightarrow & \mathsf{H}_{X_A^+} \\ \downarrow{\scriptstyle\psi^\sharp} & & \downarrow{\scriptstyle\psi^\flat} \\ H^1(\mathcal{G}_{X_B^+}) & \longrightarrow & \mathsf{H}_{X_B^+} \end{array}$$

preserves positive cones, and allows the choice of h, k, l, k', l' as in the key proposition.

Theorem

Whenever $\mathcal{G}_{X_A^+}$ and $\mathcal{G}_{X_B^+}$ are isomorphic, an isomorphism $\psi : \mathcal{G}_{X_A^+} \to \mathcal{G}_{X_B^+}$ satisfying (†) for every eventually periodic x may be chosen.

Conclusion

$$
\begin{array}{ccccc}
 & & \begin{array}{c}(\mathsf{FK}(\mathcal{O}_A),[1_{\mathcal{O}_A}])\\ \simeq \\ (\mathsf{FK}(\mathcal{O}_B),[1_{\mathcal{O}_B}])\end{array} & \Longleftrightarrow & \mathcal{O}_A \simeq \mathcal{O}_B \\
 & & & & \Uparrow \\
X_A^+ \simeq X_B^+ & \Longrightarrow & X_A^+ \sim_{\text{COOE}} X_B^+ & \Longleftrightarrow & (\mathcal{O}_A, \mathcal{C}_A) \simeq (\mathcal{O}_B, \mathcal{C}_B) \\
\Downarrow & & \Downarrow & & \Downarrow \\
X_A \simeq X_B & \Longrightarrow & X_A \sim_{\text{FE}} X_B & \Longleftrightarrow & \begin{array}{c}(\mathcal{O}_A \otimes \mathbb{K}, \mathcal{C}_A \otimes c_0)\\ \simeq \\ (\mathcal{O}_B \otimes \mathbb{K}, \mathcal{C}_B \otimes c_0)\end{array} \\
 & & & & \Downarrow \\
 & & \mathsf{FK}(\mathcal{O}_A) \simeq \mathsf{FK}(\mathcal{O}_B) & \Longleftrightarrow & \mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K}
\end{array}
$$