Group Actions and the Gromov-Hausdorff propinquity

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Noncommutative Metric Geometry

Founding Allegory of Noncommutative Metric Geometry

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- Pioneered by *Rieffel (1998–)*, inspired by *Connes (1989)*.
- Motivated by mathematical physics, addresses problems such as:
  - Can we approximate quantum spaces with finite dimensional algebras?
  - Are certain functions from a topological space to quantum spaces continuous? Lipschitz?
  - Are certain functions from a topological space to modules over quantum spaces continuous?

Structure of the talk

1. Quantum Metric Spaces From Groups

2. Group Actions and Limits for the Propinquity

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\mathsf{L}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{d}(x,y)} : x, y \in X, x \neq y \right\}$$

for all $f \in C(X)$ (allowing ∞).

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The *Monge-Kantorovich metric* on $\mathscr{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

$$\mathsf{mk}_{\mathsf{L}}(\mu,\nu) = \sup\left\{ \left| \int_X f\,d\mu - \int_X f\,d\nu \right| : f \in \mathfrak{sa}(C(X)), \mathsf{L}(f) \leqslant 1 \right\}.$$

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The Gelfand map $x \in (X, \mathsf{m}) \mapsto \delta_x \in (\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}})$ is an isometry.

Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

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We call L an *L-seminorm*.

Lipschitz morphisms

Theorem (L., 2016)

Let $(\mathfrak{A}, \mathsf{L})$ be a quasi-Leibniz quantum compact metric space. If S is a lower semi-continuous seminorm on $\mathrm{dom}(\mathsf{L})$ with $\mathrm{S}(1_{\mathfrak{A}}) = 0$ then there exists $C > 0$ such that $\mathrm{S} \leqslant C\mathsf{L}$ on $\mathrm{dom}(\mathsf{L})$.

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1. $\|\cdot\|_{\mathsf{L}} = \|\cdot\|_{\mathfrak{A}} + \mathsf{L}$ and $\|\cdot\|_{\mathsf{S}} = \|\cdot\|_{\mathfrak{A}} + \mathsf{S}$ are both Banach norms over $\mathrm{dom}(\mathsf{L})$.
2. $\|\cdot\|_{*} = \|\cdot\|_{\mathsf{L}} + \|\cdot\|_{\mathsf{S}}$ is also a Banach norm on $\mathrm{dom}(\mathsf{L})$, since Cauchy sequences for both $\|\cdot\|_{\mathsf{L}}$ and $\|\cdot\|_{\mathsf{S}}$ have the same limits for both.
3. By the open mapping theorem, there exists $k > 0$ such that $\|\cdot\|_{*} \leqslant k\|\cdot\|_{\mathsf{L}}$ so $\|\cdot\|_{\mathsf{S}} \leqslant k\|\cdot\|_{\mathsf{L}}$, and symmetrically. So $\|\cdot\|_{\mathsf{L}}$ and $\|\cdot\|_{\mathsf{S}}$ are equivalent.
4. Last, we use the fact that as L is an L-seminorm, for all $a \in \mathrm{dom}(\mathsf{L})$, there exists $t \in \mathbb{R}$ such that $\|a - t1_{\mathfrak{A}}\|_{\mathfrak{A}} \leqslant \mathrm{diam}(\mathfrak{A}, \mathsf{L})\mathsf{L}(a)$.

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Theorem-Definition (Lipschitz Morphisms)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *k-Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that any of the following equivalent statement holds:

1. $\varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$ is a *k-Lipschitz map from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ to $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$*,
2. (Rieffel, 00) $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant k\mathsf{L}_{\mathfrak{A}}$,
3. (L., 16) $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

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Lipschitz morphisms are arrows for a category over quasi-Leibniz quantum compact metric spaces.

Quantum Isometries

A real-valued Lipschitz function over a subspace of a metric space (X,d) can be extended to a real-valued Lipschitz function over (X,d) with the same Lipschitz seminorm by McShane theorem. From this, we can characterize isometries.

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Definition (Rieffel (98), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that:

$$\forall b \in \operatorname{dom}(\mathsf{L}_{\mathfrak{B}}) \quad \mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.$$

A *full quantum isometry* π is a *-isomorphism such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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By definition, quantum isometries are indeed 1-Lipschitz. They are morphisms for a subcategory of quasi-Leibniz quantum compact metric spaces.

The Gromov-Hausdorff Distance

Definition

For any two compact metric spaces (X, m_X) and (Y, m_Y), we define $\mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y)$ as:

$$\left\{ (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \;\middle|\; \begin{array}{l} (Z, \mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\}$$

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Notation

The *Hausdorff distance* on the compact subsets of a metric space (X, m) is denoted by $\mathsf{Haus}_{\mathsf{m}}$.

Definition (Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X, m_X) and (Y, m_Y) is:

$$\inf \left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X), \iota_Y(Y)) : (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \in \mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y) \right\}.$$

The Dual Gromov-Hausdorff Propinquity

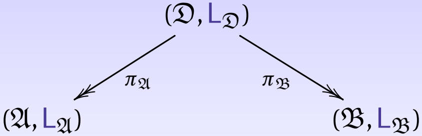

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

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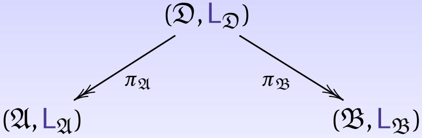

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***Definition** (The extent of a tunnel)*

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

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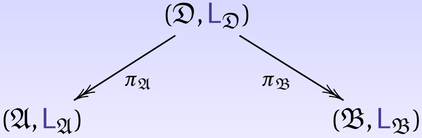

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Definition (L. ,13, 14 / special case)

The *dual propinquity* $\Lambda_F^{*}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

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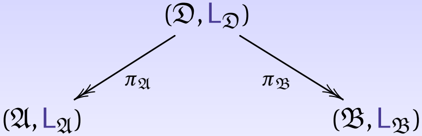

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Theorem (L., 13)

The dual propinquity is a *complete metric* up to *full quantum isometry*: $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ iff there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

Classical Picture

Example: Gromov-Hausdorff distance (L. 13)

The *dual propinquity* induces the same topology as the *Gromov-Hausdorff distance* on the class of classical compact metric spaces $(C(X), \mathsf{L})$ with (X, d) compact metric space and:

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A recurrent theme

Groups and in particular, group actions provide good tools.

Ergodic actions of compact metric groups

Theorem (Rieffel, 98)

Let G be a *compact group* endowed with a *continuous length function* ℓ. Let α be an *action* of G on some *unital C*-algebra* $\mathfrak{A}$. Set:

$$\forall a \in \mathfrak{A} \quad \mathsf{L}(a) = \sup\left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1_G\} \right\}.$$

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Example: Quantum tori

- $G = \mathbb{T}^d$,
- $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$ (universal for $U_j U_k = \sigma(j,k) U_{j+k}$),
- α : dual action ($\alpha^z U_j = z^j U_j$).
- Associated with a differential calculus when ℓ from invariant Riemannian metric.

Idea of the proof

Let λ be the Haar probability measure on G.

1. For $f \in L^1(G)$ and $a \in \mathfrak{A}$ set:

$$\alpha^f(a) = \int_G f(g)\alpha^g(a)\,d\lambda(g).$$

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4. Now:
$$\left\|\alpha^f(a) - a\right\|_{\mathfrak{A}} \leqslant \left(\int_G |f(g)|\ell(g)\,d\lambda(g)\right)\mathsf{L}(a).$$
We can choose an approximate unit $(f_n)_{n\in\mathbb{N}}$ in $L^1(G)$ made of finite linear combinations of irreducible characters. Then $\left(\int_G f_n \ell\,d\lambda\right)_{n\in\mathbb{N}}$ converges to 0.

Quantum Tori and the quantum propinquity

Endow $\mathbb{T}^d$ with a continuous length function ℓ. Let α be the dual action of $\widehat{\mathbb{Z}_k} \subseteq \mathbb{T}^d$ on $C^*(\mathbb{Z}^d_k, \sigma)$, and set for $a \in \mathfrak{sa}\left(\mathbb{C}^*(\mathbb{Z}^d_k, \sigma)\right)$:

$$\mathsf{L}_\sigma(a) = \sup\left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in \widehat{\mathbb{Z}^d_k} \setminus \{1\} \right\}.$$

L_σ is an L-seminorm (Rieffel, 98).

Theorem (Latrémolière, 2013)

Let $d \in \mathbb{N} \setminus \{0,1\}$, σ a multiplier of $\mathbb{Z}^d$. If for each $n \in \mathbb{N}$, we let $k_n \in \overline{\mathbb{N}}^d$ and σ_n be a multiplier of $\mathbb{Z}^d_k = \mathbb{Z}^d / k_n \mathbb{Z}^d$ such that:

1. $\lim_{n\to\infty} k_n = (\infty, \ldots, \infty)$,
2. the unique lifts of σ_n to $\mathbb{Z}^d$ as multipliers converge pointwise to σ,

then $\lim_{n\to\infty} \Lambda\left(\left(C^*\left(\mathbb{Z}^d, \sigma\right), \mathsf{L}_\sigma\right), \left(C^*\left(\mathbb{Z}^d_{k_n}, \sigma_n\right), \mathsf{L}_{\sigma_n}\right)\right) = 0.$

Noncommutative Solenoids

Let $\mathbb{Z}\left[\frac{1}{p}\right] = \left\{\frac{q}{p^k} : q \in \mathbb{Z}, k \in \mathbb{N}\right\} \subseteq \mathbb{Q}$. The dual of $\mathbb{Z}\left[\frac{1}{p}\right]$ is:

$$\mathscr{S}_p = \left\{(z_n)_{n\in\mathbb{N}} : \forall n \in \mathbb{N} \quad z_n = z_{n+1}^p\right\}$$

with pairing $\langle \frac{q}{p^k}, (z_n)_{n\in\mathbb{N}} \rangle = z_k^q$.

Theorem (J. Packer, L. 2011)

Any multiplier f of $\mathbb{Z}\left[\frac{1}{p}\right]^2$ is cohomologous to one of the form:

$$\psi_\theta\left(\left(\frac{p_j}{q^{k_j}}\right)_{j=1}^4\right) = \theta_{k_1+k_4}^{q_1 q_4}$$

where $\theta \in \mathscr{S}_p$ is uniquely determined by f.

The twisted C*-algebra $C^*\left(\mathbb{Z}\left[\frac{1}{p}\right]^2, \psi_\theta\right)$ is the noncommutative solenoid $\mathfrak{S}_\theta$.

Noncommutative Solenoids and the Propinquity

Pick ℓ a continuous length function on $\mathbb{T}^2$. For $\eta \in \mathscr{S}_p^2 \subseteq (\mathbb{T}^2)^{\mathbb{N}}$, define:

$$\ell_\infty(\eta) = \inf\left\{\varepsilon > 0 : \forall n < \frac{\operatorname{diam}\left(\mathbb{T}^2, \ell\right)}{\varepsilon} \quad \ell(\eta_n) < \varepsilon\right\}.$$

Theorem (J. Packer, L., 2016)

The function ℓ_∞ is a continuous length function on $\mathscr{S}_p^2$. For all $n \in \mathbb{N}$ and $z \in \mathbb{T}^2$, let:

$$\ell_n(z) = \inf\left\{\ell_\infty(\eta) : \eta_n = z\right\}.$$

Then ℓ_n is a continuous length function on $\mathbb{T}^2$. If $\theta \in \mathscr{S}_p$ then:

$$\lim_{n\to\infty} \Lambda^*\left(\mathfrak{S}_\theta, C^*(\mathbb{Z}^2, \theta_{2n})\right) = 0.$$

In particular, noncommutative solenoids are limits of full matrix algebras for the propinquity and form a continuous family.

A metric on the group of automorphisms

Theorem

If $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space and $\alpha \in \mathrm{Aut}(\mathfrak{A})$ then, by setting:

$$\sup\{\|a - \alpha(a)\|_{\mathfrak{A}} : \mathsf{L}(a) \leqslant 1\}$$

we define a length function on $\mathrm{Aut}(\mathfrak{A})$ which metrizes the pointwise convergence topology.

Group Gromov-Hausdorff convergence

Let (G_1,δ_1), (G_2,δ_2) be two compact metric groups.

Definition (L., 2017)

A *ε-unital equivariant isometry* (ξ_1,ξ_2) from (G_1,δ_1) to (G_2,δ_2) is an ordered pair such that for all $\{j,k\}=\{1,2\}$:

1. $\xi_j : G_j \to G_k$,
2. $\forall g,g' \in G_j \quad |\delta_k(\xi_j(g),\xi_j(g')) - \delta_j(g,g')| \leqslant \varepsilon$,
3. $\forall g \in G_j \quad \delta_j(\xi_k \circ \xi_j(g), g) \leqslant \varepsilon$,
4. $\forall g,g' \in G_j \quad \delta_k(\xi_j(gg'),\xi_j(g)\xi_j(g')) \leqslant \varepsilon$,
5. $\delta_j(e_j,\xi_k(e)) \leqslant \varepsilon$.

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3. $\forall g \in G_j \quad \delta_j(\xi_k \circ \xi_j(g), g) \leqslant \varepsilon$,
4. $\forall g, g' \in G_j \quad \delta_k(\xi_j(gg'), \xi_j(g)\xi_j(g')) \leqslant \varepsilon$,
5. $\delta_j(e_j, \xi_k(e)) \leqslant \varepsilon$.

Definition (L., 2017)

The *equivariant Gromov-Hausdorff distance* $\Upsilon((G_1, \delta_1), (G_2, \delta_2))$ between (G_1, δ_1) and (G_2, δ_2) is:

$$\inf\left\{\varepsilon > 0 : \exists (\xi_1, \xi_2) \quad \varepsilon\text{-unital quasi isometry } (G_1, \delta_1) \to (G_2, \delta_2)\right\}$$

Group Gromov-Hausdorff convergence

Let (G_1,δ_1), (G_2,δ_2) be two compact metric groups.

Definition (L., 2017)

A *ε-unital equivariant isometry* (ξ_1,ξ_2) from (G_1,δ_1) to (G_2,δ_2) is an ordered pair such that for all $\{j,k\}=\{1,2\}$:

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3. $\forall g \in G_j \quad \delta_j(\xi_k \circ \xi_j(g), g) \leqslant \varepsilon$,
4. $\forall g,g' \in G_j \quad \delta_k(\xi_j(gg'),\xi_j(g)\xi_j(g')) \leqslant \varepsilon$,
5. $\delta_j(e_j,\xi_k(e)) \leqslant \varepsilon$.

Theorem (L., 2017)

Υ is a metric on compact metric groups up to isometric isomorphism, and it dominates the Gromov-Hausdorff distance.

Theorem (L., 2017)

Let (G,δ_G) be a compact metric group, and let $(G_n,\delta_n)_{n\in\mathbb{N}}$ be a sequence of compact metric groups converging to (G,δ_G). For each $n\in\mathbb{N}$, let (ς_n,χ_n) be a $\Upsilon(G_n,G)$-unital equivariant isometry. Let $D : G \to [0,\infty)$ be a continuous function and $K : [0,\infty) \to [0,\infty)$ be continuous at 0 with $K(0) = 0$.

Let $(\mathfrak{A}_n,\mathsf{L}_n)_{n\in\mathbb{N}}$ converge to $(\mathfrak{A},\mathsf{L})$ for Λ^*. If for all $n\in\mathbb{N}$, there exists an action α_n of G_n on $\mathfrak{A}_n$ such that:

$$\forall n\in\mathbb{N}, g\in G_n \quad \mathsf{L}_n\circ\alpha_n^g \leqslant D(\varsigma_n(g))\mathsf{L}_n$$
$$\forall n\in\mathbb{N}, g,g'\in G_n, a\in\mathfrak{A}_n \quad \left\|\alpha_n^g(a)-\alpha_n^{g'}(a)\right\|_{\mathfrak{A}} \leqslant K(\delta_n(g,g'))\|a\|_{\mathfrak{A}}$$

then there exists a strongly continuous action α of G on $\mathfrak{A}$ such that:

$$\forall g\in G \quad \mathsf{L}\circ\alpha^g \leqslant D(g)\mathsf{L}$$
$$\liminf_{n\to\infty}\mathsf{mk}\ell_{\mathsf{L}_n}(\alpha_n^{\kappa_n(g)}) \leqslant \mathsf{mk}\ell_{\mathsf{L}}(\alpha^g) \leqslant \limsup_{n\to\infty}\mathsf{mk}\ell_{\mathsf{L}_n}(\alpha_n^{\kappa_n(g)}).$$

Inductive Limits

Theorem

Let $G = \bigcup_{n\in\mathbb{N}} G_n$ be a group with G_n a first countable subgroup of G, and endow G with the inductive limit topology. Assume that for all $n \in \mathbb{N}$, there is an action α_n of G_n on some quasi-Leibniz quantum compact metric space $(\mathfrak{A}_n, \mathsf{L}_n)$ with:

$$\mathsf{L}_n \circ \alpha_n^g \leqslant D(g)\mathsf{L}_n$$
$$\|\alpha_n^g(a) - \alpha_n^h(a)\|_{\mathfrak{A}_n} \leqslant K(g,h)\|a\|_{\mathfrak{A}_n},$$

for $D : G \to [0,\infty)$ and $K : G^2 \to [0,\infty)$ continuous, $K(h,g) = 0$ for all $g \in G$.

If $(\mathfrak{A}_n, \mathsf{L}_n)_{n\in\mathbb{N}}$ converges for the propinquity to $(\mathfrak{A}, \mathsf{L})$, then there exists an action α of G on $\mathfrak{A}$ such that:

$$\forall g \in G \quad \mathsf{L} \circ \alpha^g \leqslant D(g)\mathsf{L}$$
$$\forall g \in G \quad \liminf_{n\to\infty} \mathsf{mk}\ell_{\mathsf{L}_n}(\alpha_n^g) \leqslant \mathsf{mk}\ell_{\mathsf{L}}(\alpha^g) \leqslant \limsup_{n\to\infty} \mathsf{mk}\ell_{\mathsf{L}_n}(\alpha_n^g).$$

Elements of the proof

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two F–quasi-Leibniz quantum compact metric spaces.

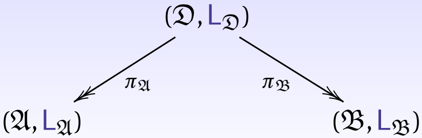

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

Definition (L., 2013)

Let $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be a tunnel. Let $a \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$. The *$l$-target set* $\mathfrak{t}_{\tau}(a|l)$ is:

$$\left\{ \pi_{\mathfrak{B}}(d) \in \mathfrak{sa}(\mathfrak{B}) \,\middle|\, \begin{array}{l} \mathsf{L}_{\mathfrak{D}}(d) \leqslant l, \\ \pi_{\mathfrak{A}}(d) = a \end{array} \right\}.$$

Elements of the proof

Definition (L., 2013)

Let $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be a tunnel. Let $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$. The *$l$-target set* $\mathfrak{t}_{\tau}(a|l)$ is:

$$\left\{ \pi_{\mathfrak{B}}(d) \in \mathfrak{sa}(\mathfrak{B}) \middle| \begin{array}{l} \mathsf{L}_{\mathfrak{D}}(d) \leqslant l, \\ \pi_{\mathfrak{A}}(d) = a \end{array} \right\}.$$

Theorem (L., 2013)

Let $a, a' \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{A}}(a')\}$. Let $b \in \mathfrak{t}_{\tau}(a|l)$ and $b' \in \mathfrak{t}_{\tau}\left(a' \middle| l\right)$.

1. $\|b - b'\|_{\mathfrak{B}} \leqslant 2l\chi(\tau) + \|a - a'\|_{\mathfrak{A}}$,
2. $\mathrm{diam}(\mathfrak{t}_{\tau}(a|l), \|\cdot\|_{\mathfrak{B}}) \leqslant 2l\chi(\tau)$,
3. $b + tb' \in \mathfrak{t}_{\tau}\left(a + ta' \middle| (1 + |t|)l\right)$ for all $t \in \mathbb{R}$,
4. $b \circ b' \in \mathfrak{t}_{\tau}\left(a \circ a' \middle| G(l, \|a\|_{\mathfrak{A}}, \|a'\|_{\mathfrak{A}}, \chi(\tau))\right)$ and same for the Lie product.

Composition of tunnels

If tunnels are approximate morphisms, can they be composed?

Thank you!