Quantum Metric Spaces

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Noncommutative Geometry

Founding Allegory of Noncommutative Geometry

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Definition (Gelfand, 1956)

A *C*-algebra* $\mathfrak{A}$ is an associative algebra over $\mathbb{C}$, endowed with a norm $\|\cdot\|_{\mathfrak{A}}$ and an antilinear, antimultiplicative involution $a \in \mathfrak{A} \mapsto a^* \in \mathfrak{A}$ such that:

1. $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}})$ is a Banach space,
2. $\forall a, b \in \mathfrak{A} \quad \|ab\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}} \|b\|_{\mathfrak{B}}$,
3. $\forall a \in \mathfrak{A} \quad \|aa^*\|_{\mathfrak{A}} = \|a\|_{\mathfrak{A}}^2$.

A C*-algebra A is *unital* when it has a multiplicative unit $1_{\mathfrak{A}}$.
A **-morphism* is an algebraic morphism for the $*$-algebra structure.

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Theorem (Gelfand Duality)

The category of Abelian unital C*-algebras is dual to the category of compact Hausdorff spaces.

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Definition (State Space)

A *state* of a C*-algebra $\mathfrak{A}$ is a positive linear functional ($\varphi(a^* a) \geqslant 0$) of norm 1. The set $\mathscr{S}(\mathfrak{A})$ of all states of $\mathfrak{A}$ is the *state space* of $\mathfrak{A}$.

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The extreme points of $\mathscr{S}(\mathfrak{A})$ are *pure states*. If $\mathfrak{A}$ is an Abelian unital C*-algebra, then $\mathfrak{A}$ is *-isomorphic to the C*-algebra of $\mathbb{C}$-valued continuous functions on the pure state space.

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Example: Noncommutative Topology

Noncommutative *topology* is the study of (unital) C*-algebras as generalizations of the algebras of *continuous* functions over a (locally) *compact* Hausdorff space.

This framework includes the development of K-theory, KK-theory, …

Other examples of this analogy includes quantum groups as Hopf algebras, noncommutative Riemannian manifolds as spectral triples, Von Neumann algebras as noncommutative measurable spaces.

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- Pioneered by *Rieffel (1998–)*, inspired by *Connes (1989)*.
- Motivated by mathematical physics, addresses problems such as:
  - Can we approximate quantum spaces with finite dimensional algebras?
  - Are certain functions from a topological space to quantum spaces continuous? Lipschitz?
  - Are certain functions from a topological space to modules over quantum spaces continuous?

Structure of the talk

1. Compact Quantum Metric Spaces
2. Convergence of quasi-Leibniz quantum compact metric space
3. Gromov's Compactness Theorem

GPS

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\mathsf{L}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)} : x, y \in X, x \neq y \right\}$$

for all $f \in C(X)$ (allowing ∞).

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The *Monge-Kantorovich metric* on $\mathscr{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

$$\mathsf{mk}_{\mathsf{L}}(\mu,\nu) = \sup\left\{\left|\int_X f\,d\mu - \int_X f\,d\nu\right| : f \in \mathfrak{sa}(C(X)), \mathsf{L}(f) \leqslant 1\right\}.$$

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The Gelfand map $x \in (X, \mathsf{m}) \mapsto \delta_x \in (\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}})$ is an isometry.

Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

1. $\mathfrak{A}$ is a *unital C*-algebra*,

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We call L an *L-seminorm*.

Lipschitz morphisms and Quantum Isometries

Theorem-Definition (Lipschitz Morphisms)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *k-Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that any of the following equivalent statement holds:

1. $\varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$ is a *k-Lipschitz map from* $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ *to* $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$,
2. (Rieffel, 00) $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant k \mathsf{L}_{\mathfrak{A}}$,
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Definition (Rieffel (98), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a $*$-epimorphism such that:

$$\forall b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) \quad \mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.$$

A *full quantum isometry* π is a $*$-isomorphism such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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Theorem (McShane, 1934)

Let (Z, m) be a metric space and $X \subseteq Z$ not empty. If $f : X \to \mathbb{R}$ is a k-Lipschitz function on X then there exists a k-Lipschitz function $g : Z \to \mathbb{R}$ such that g restricts to f on X.

Lexicon

Classical world	Quantum world

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Compact Hausdorff Space	Unital C*-algebra

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Isometry	$\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \twoheadrightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ with $\mathsf{L}_{\mathfrak{B}} = \inf \mathsf{L}_{\mathfrak{A}}\left(\pi^{-1}(\cdot)\right)$.

AF algebras with tracial state

Theorem (Aguilar, L., 15)

- Let $\mathfrak{A} = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$ be an *AF-algebra with a faithful tracial state* τ and where $\dim \mathfrak{A}_n < \infty$ for all $n \in \mathbb{N}$, $\mathfrak{A}_0 = \mathbb{C}$.
- For all $n \in \mathbb{N}$ let $\mathbb{E}_n : \mathfrak{A} \to \mathfrak{A}_n$ be the unique conditional expectation with $\tau \circ \mathbb{E}_n = \tau$.
- Let $(\beta_n)_{n\in\mathbb{N}}$ in $(0,\infty)^{\mathbb{N}}$, with limit 0.

If, for all $a \in \mathfrak{sa}(\mathfrak{A})$, we set:

$$\mathsf{L}(a) = \sup\left\{ \frac{\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}}}{\beta_n} : n \in \mathbb{N} \right\}$$

then $(\mathfrak{A}, \mathsf{L})$ is a $(2,0)$–quasi-Leibniz quantum compact metric space.

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The quasi-Leibniz condition here is:

$$\forall a,b \in \mathfrak{A} \quad \max\{\mathsf{L}(a \circ b), \mathsf{L}(\{a,b\})\} \leqslant 2(\mathsf{L}_{\mathfrak{A}}(a)\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}}\mathsf{L}_{\mathfrak{A}}(b)).$$

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This construction recovers the usual ultrametrics of the Cantor set.

Ergodic actions of compact metric groups

Theorem (Rieffel, 98)

Let G be a *compact group* endowed with a *continuous length function ℓ*. Let α be an *action* of G on some *unital C*-algebra $\mathfrak{A}$*. Set:

$$\forall a \in \mathfrak{A} \quad \mathsf{L}(a) = \sup\left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1_G\} \right\}.$$

$(\mathfrak{A}, \mathsf{L})$ is a *Leibniz quantum compact metric space* if and only if $\left\{a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a\right\} = \mathbb{C}1_{\mathfrak{A}}$.

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Example: Quantum tori

- $G = \mathbb{T}^d$,
- $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$ (universal for $U_j U_k = \sigma(j,k) U_{j+k}$),
- α : dual action ($\alpha^z U_j = z^j U_j$).
- Associated with a differential calculus when ℓ from invariant Riemannian metric.

Spectral triples and quantum metrics

Theorem (L., 15)

Let α be an *ergodic action* of a *compact Lie group G* on a unital C*-algebra $\mathfrak{A}$. Let π be the GNS rep from the invariant tracial state of $\mathfrak{A}$. Let $(X_1,\ldots,X_n)$ be an orthonormal basis for the *Lie algebra* $\mathfrak{g}$ of G equipped with an inner product, and let:

$$\partial_j : a \in \mathfrak{A}^1 \mapsto \lim_{t\to 0} \frac{\alpha^{\exp(tX_j)} a - a}{t} \text{ for } j \in \{1,\ldots,n\}.$$

For $H = \begin{pmatrix} h_{11} & \ldots & h_{1n} \\ \vdots & & \vdots \\ h_{n1} & \ldots & h_{nn} \end{pmatrix} \in M_n(\pi(\mathfrak{A})')$, we set:

$$D_H = \sum_{j=1}^{n} \sum_{k=1}^{n} h_{jk}\partial_k \otimes \mathrm{Clifford}(X_j)$$

$\left(\mathfrak{A}, \left|\left|\left|[D_H,\pi^{\oplus n}(\cdot)]\right|\right|\right|\right)$ is a *Leibniz quantum compact metric space*.

Some Other Examples

1. Hyperbolic group C*-algebras (Rieffel, Ozawa, 05),
2. Nilpotent group C*-algebras (Christ, Rieffel, 16),
3. Connes-Landi spheres (Li, 03)
4. Conformal deformations of quantum metric spaces from spectral triples (L., 15)
5. Group C*-algebras for groups with rapid decay (Antonescu, Christensen, 2004)
6. Noncommutative Solenoids (L., Packer, 16)
7. Certain C*-crossed-products (J. Bellissard, M. Marcolli, Reihani, 10), (involves my work on locally compact quantum metric space).

GPS

The Gromov-Hausdorff Distance

Definition

For any two compact metric spaces (X, m_X) and (Y, m_Y), we define $\mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y)$ as:

$$\left\{ (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \;\middle|\; \begin{array}{l} (Z, \mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\}$$

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Notation

The *Hausdorff distance* on the compact subsets of a metric space (X, m) is denoted by $\mathsf{Haus}_{\mathsf{m}}$.

Definition (Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X, m_X) and (Y, m_Y) is:

$$\inf\left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X), \iota_Y(Y)) : (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \in \mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y) \right\}.$$

The Dual Gromov-Hausdorff Propinquity

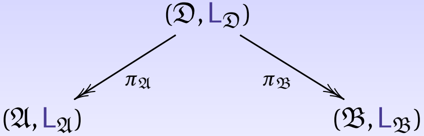

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

The Dual Gromov-Hausdorff Propinquity

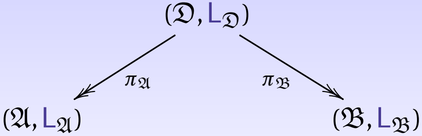

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

Definition (The extent of a tunnel)

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{A}))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}(\mathscr{S}(\mathfrak{B}))\right)\right\}.$$

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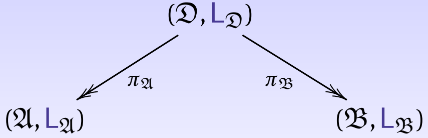

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Definition (L. ,13, 14 / special case)

The *dual propinquity* $\Lambda_F^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\inf\left\{\chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.$$

The Dual Gromov-Hausdorff Propinquity

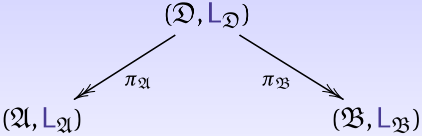

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

Definition (L., 13, 14 / special case)

The *dual propinquity* $\Lambda^*_F((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\inf\left\{\chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.$$

Theorem (L., 13)

The dual propinquity is a *complete metric* up to *full quantum isometry*, which induces the same topology on classical compact metric spaces as the Gromov-Hausdorff distance.

Quantum Tori and the quantum propinquity

Endow $\mathbb{T}^d$ with a continuous length function ℓ. Let α be the dual action of $\widehat{\mathbb{Z}_k} \subseteq \mathbb{T}^d$ on $C^*(\mathbb{Z}_k^d, \sigma)$, and set for $a \in \mathfrak{sa}\left(\mathbb{C}^*(\mathbb{Z}_k^d,\sigma)\right)$:

$$\mathsf{L}_\sigma(a) = \sup\left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in \widehat{\mathbb{Z}_k^d} \setminus \{1\} \right\}.$$

L_σ is an L-seminorm (Rieffel, 98).

Theorem (Latrémolière, 2013)

Let $d \in \mathbb{N} \setminus \{0,1\}$, σ a multiplier of $\mathbb{Z}^d$. If for each $n \in \mathbb{N}$, we let $k_n \in \overline{\mathbb{N}}^d$ and σ_n be a multiplier of $\mathbb{Z}_k^d = \mathbb{Z}^d / k_n\mathbb{Z}^d$ such that:

1. $\lim_{n\to\infty} k_n = (\infty, \ldots, \infty)$,
2. the unique lifts of σ_n to $\mathbb{Z}^d$ as multipliers converge pointwise to σ,

then $\lim_{n\to\infty} \Lambda\left(\left(C^*\left(\mathbb{Z}^d,\sigma\right), \mathsf{L}_\sigma\right), \left(C^*\left(\mathbb{Z}_{k_n}^d, \sigma_n\right), \mathsf{L}_{\sigma_n}\right)\right) = 0.$

Curved Quantum Tori

Theorem (L., 15)

- Let G be a compact Lie group acting ergocially on a unital C*-algebra $\mathfrak{A}$. Let τ be the G-invariant tracial state of $\mathfrak{A}$.
- Identify $\mathfrak{A}$ with its image by the regular representation acting on $L^2(\mathfrak{A},\tau)$ (τ canonical trace)
- Define the length function $\ell : K \in \mathrm{GL}_d(\mathfrak{A}') \mapsto |||1 - K|||_{L^2(\mathfrak{A},\tau)}$.

If $H \in \mathrm{GL}_d(\mathfrak{A}')$ then:

$$\lim_{\substack{G\to H\\ G\in\mathfrak{GL}_n(\mathfrak{A}')}} \Lambda((\mathfrak{A},\mathsf{L}_G),(\mathfrak{A},\mathsf{L}_H)) = 0$$

where $\mathsf{L}_H = |||[D_H,\cdot]|||$ with $D_H = \sum_{j=1}^n \sum_{k=1}^n H_{jk}\partial_k \otimes X_j$ for a fixed orthonormal basis $(X_1,\ldots,X_d)$ of the Lie algebra of G, equipped with some inner product.

Effros-Shen AF algebras

Theorem (Aguilar, L., 15)

- For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $\theta = \lim_{n\to\infty} \frac{p_n^\theta}{q_n^\theta}$ with $\frac{p_n^\theta}{q_n^\theta} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}$ for $a_1, \ldots \in \mathbb{N}$.
- Set $\mathfrak{AF}_\theta = \varinjlim_{n\to\infty} \left(\mathfrak{M}_{q_n} \oplus \mathfrak{M}_{q_{n-1}}, \psi_{n,\theta}\right)$ where $\psi_{n,\theta}$ involves a_{n+1}.
- For all $n \in \mathbb{N}$, let $\beta_n = \frac{1}{q_n^2 + q_{n-1}^2}$ and L_θ the L-seminorm for this data.

For all $\theta \in \mathbb{R} \setminus \mathbb{Q}$, we have:

$$\lim_{\substack{\vartheta\to\theta \\ \vartheta\notin\mathbb{Q}}} \Lambda\left((\mathfrak{AF}_\vartheta, \mathsf{L}_\vartheta), (\mathfrak{AF}_\theta, \mathsf{L}_\theta)\right) = 0.$$

GPS

Gromov's compactness Theorem

What is a noncommutative analogue of the following?

Theorem (Gromov, 1981)

A class $\mathscr{S}$ of compact metric spaces is totally bounded for the Gromov-Hausdorff distance if, and only if the following two assertions hold:

1. there exists $D \geqslant 0$ such that for all $(X, \mathsf{m}) \in \mathscr{S}$, the diameter of (X, m) is less or equal to D,
2. there exists a function $G : (0, \infty) \rightarrow \mathbb{N}$ such that for every $(X, \mathsf{m}) \in \mathscr{S}$, and for every $\varepsilon > 0$, the smallest number $\mathrm{Cov}_{(X,\mathsf{m})}(\varepsilon)$ of balls of radius ε needed to cover (X, m) is no more than $G(\varepsilon)$.

Since the Gromov-Hausdorff distance is complete, a class of compact metric spaces is compact for the Gromov-Hausdorff distance if and only if it is closed and totally bounded.

A Compactness Theorem for Λ_F

Definition (Latrémolière, 2015)

Let F be a permissible function and $\mathcal{Q}$ the class of F–quasi-Leibniz quantum compact metric spaces. If $(\mathfrak{A}, \mathsf{L}) \in \mathcal{Q}$ and $\varepsilon > 0$ then we define the covering number $\mathrm{cov}_{(F)}(\mathfrak{A}, \mathsf{L}|\varepsilon)$:

$$\min\left\{\dim \mathfrak{B} : (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{Q}, \Lambda_F((\mathfrak{A}, \mathsf{L}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \varepsilon\right\}.$$

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Theorem (Latrémolière, 2015)

Let F be a continuous, permissible function. Let $\mathscr{F}$ be the closure of the class of *finite dimensional* F–quasi-Leibniz quantum compact metric spaces for Λ_F. A subclass $\mathscr{A}$ of $\mathscr{F}$ is *totally bounded* for Λ_F if and only if there exists $G : [0, \infty) \to \mathbb{N}$ and $K \geqslant 0$ such that for all $(\mathfrak{A}, \mathsf{L}) \in \mathscr{A}$:

- $\mathrm{diam}\left(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}}\right) \leqslant K$,
- $\mathrm{cov}_{(F)}(A, \mathsf{L}|\varepsilon) \leqslant G(\varepsilon)$.

What can we find in the closure of f.d. quasi-Leibniz quantum compact metric spaces?

Definition (Latrémolière, 2015)

A unital C*-algebra $\mathfrak{A}$ is *pseudo-diagonal* when, for all $\varepsilon > 0$ and for all finite subset $\mathfrak{F}$ of $\mathfrak{A}$, there exists a finite dimensional C*-algebra $\mathfrak{B}$ and two unital, positive linear maps $\varphi : \mathfrak{B} \to \mathfrak{A}$ and $\psi : \mathfrak{A} \to \mathfrak{B}$ such that for all $a, b \in \mathfrak{F}$:

- $\|a - \varphi \circ \psi(a)\|_{\mathfrak{A}} \leqslant \varepsilon$,
- $\|\psi(a)\psi(b) - \psi(ab)\|_{\mathfrak{B}} \leqslant \varepsilon$.

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Theorem (Latrémolière, 2015)

Every unital, nuclear, quasi-diagonal C*-algebra is pseudo-diagonal.

This uses a result from Blackadar and Kirchberg.

What can we find in the closure of f.d. quasi-Leibniz quantum compact metric spaces?

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- $\|\psi(a)\psi(b) - \psi(ab)\|_{\mathfrak{B}} \leqslant \varepsilon$.

Notation (Latrémolière, 2015)

If $C \geqslant 1$, $D \geqslant 0$, a quasi-Leibniz quantum compact metric space $(\mathfrak{A}, \mathsf{L})$ such that for all $a, b \in \mathfrak{sa}(\mathfrak{A})$:

$$\mathsf{L}(a \circ b) \vee \mathsf{L}(\{a, b\}) \leqslant C\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{B}}\right) + D\mathsf{L}(a)\mathsf{L}(b)$$

is a (C, D)–quasi-Leibniz quantum compact metric space.

What can we find in the closure of f.d. quasi-Leibniz quantum compact metric spaces?

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A unital C*-algebra $\mathfrak{A}$ is *pseudo-diagonal* when, for all $\varepsilon > 0$ and for all finite subset $\mathfrak{F}$ of $\mathfrak{A}$, there exists a finite dimensional C*-algebra $\mathfrak{B}$ and two unital, positive linear maps $\varphi : \mathfrak{B} \to \mathfrak{A}$ and $\psi : \mathfrak{A} \to \mathfrak{B}$ such that for all $a, b \in \mathfrak{F}$:

- $\|a - \varphi \circ \psi(a)\|_{\mathfrak{A}} \leqslant \varepsilon$,
- $\|\psi(a)\psi(b) - \psi(ab)\|_{\mathfrak{B}} \leqslant \varepsilon$.

Theorem (Latrémolière, 2015)

Let $C \geqslant 1$, $D \geqslant 0$. If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is a (C, D)–quasi-Leibniz quantum compact metric space with $\mathfrak{A}$ pseudo-diagonal, and if $\delta > 0$, then there exists a sequence $(\mathfrak{B}_n, \mathsf{L}_n)_{n\in\mathbb{N}}$ of *finite dimensional $(C+\delta, D+\delta)$–quasi-Leibniz quantum compact metric spaces* such that $\lim_{n\to\infty} \Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}_n, \mathsf{L}_n)) = 0$.

Other examples

1. Conformal perturbations of quantum metrics (L., 15)
2. AF algebras as limits of their inductive sequence in a *metric* sense; UHF and Effros-Shen algebras form continuous families (Aguilar and L., 15),
3. Spheres as limits of full matrix algebras (Rieffel, 15)
4. Nuclear quasi-diagonal quasi-Leibniz quantum compact metric spaces have finite dim approximations (L., 15),
5. There exists an analogue of Gromov's compactness theorem (L., 15)
6. Noncommutative solenoids form a continuous family and have approximations by quantum tori (L. and Packer, 16)
7. Closed balls for the noncommutative Lipschitz distance are totally bounded for Λ (L., 16)

Thank you!

- *The Quantum Gromov-Hausdorff Propinquity*, F. Latrémolière, *Transactions of the AMS* **368** (2016) 1, pp. 365–411, ArXiv: 1302.4058
- *Convergence of Fuzzy Tori and Quantum Tori for the quantum Gromov-Hausdorff Propinquity: an explicit Approach*, F. Latrémolière, *Münster Journal of Mathematics* **8** (2015) 1, pp. 57–98, ArXiv: 1312.0069
- *The Dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Journal de Mathématiques Pures et Appliquées* **103** (2015) 2, pp. 303–351, ArXiv: 1311.0104
- *A compactness theorem for the dual Gromov-Hausdorff Propinquity*, F. Latrémolière, Accepted in *Indiana University Mathematics Journal* (2015), 40 pages, Arxiv: 1501.06121
- *Quantum Ultrametrics on AF algebras*, K. Aguilar and F. Latrémolière, *Studia Math.* **231** (2015) 2, pp. 149–193, ArXiv: 1511.07114
- *The Modular Gromov-Hausdorff Propinquity*, F. Latrémolière, Submitted (2015), 130 pages, ArXiv: 1608.04881, 1703.07073