Convergence for Quantum Compact Metric Spaces and their Modules

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A question in noncommutative geometry

Is there a sensible meaning to the intuitive statement that:

$$C^*\left(U_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & \dots & 1 & 0 \end{pmatrix}, V_n = \begin{pmatrix} 1 & & & & \\ & \rho_n & & & \\ & & \rho_n^2 & & \\ & & & \ddots & \\ & & & & \rho_n^{n-1} \end{pmatrix} \right)$$

with $\rho_n = \exp\left(\frac{2i\pi p_n}{n}\right) \neq 1$, converges, as $n \to \infty$ and $\rho_n \to \exp(2i\pi\theta)$, to the universal C*-algebra $\mathscr{A}_\theta = C^*(U,V)$ for the relations:

$$UV = \exp(2i\pi\theta)VU \text{ and } U^* = U^{-1}, V^* = V^{-1}?$$

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An idea is to introduce the notion of a *quantum metric space* and employ a generalization of the *Gromov-Hausdorff distance*.

A deeper question

For all $\theta \in \mathbb{R}$, let $\mathscr{A}_\theta$ be the universal C*-algebra for the relations:

$$UV = \exp(2i\pi\theta)VU \text{ and } U^* = U^{-1}, V^* = V^{-1}.$$

Suppose now that $\theta \in \mathbb{R} \mapsto \mathscr{A}_\theta$ is *continuous* in the sense of some *topology*, as per our first question.

Let $p, q \in \mathbb{Z} \setminus \{0\}$. For all θ, let $\mathscr{M}_\theta$ be the *module* over $\mathscr{A}_\theta$ whose class in $K_0(\theta)$ has trace $q\theta - p$.

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An idea is to extend the *Gromov-Hausdorff topology* not only to the noncommutative realm, but also to some classes of *Hilbert modules*.

Structure of the talk

1. Compact Quantum Metric Spaces
2. Convergence of quasi-Leibniz quantum compact metric space
3. The Modular Propinquity

GPS

Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

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$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\left\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}), \mathsf{L}(a) \leqslant 1\right\}$$

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We call L an *L-seminorm*.

Lipschitz morphisms and Quantum Isometries

Theorem-Definition (Lipschitz Morphisms)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *k-Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that any of the following equivalent statement holds:

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2. (Rieffel, 00) $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant k \mathsf{L}_{\mathfrak{A}}$,
3. (L., 16) $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

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Definition (Rieffel (98), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that:

$$\forall b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) \quad \mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.$$

A *full quantum isometry* π is a *-isomorphism such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

Ergodic actions of compact metric groups

Theorem (Rieffel, 98)

Let G be a *compact group* endowed with a *continuous length function* ℓ. Let α be an *action* of G on some *unital C*-algebra* $\mathfrak{A}$. Set:

$$\forall a \in \mathfrak{A} \quad \mathsf{L}(a) = \sup\left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1_G\} \right\}.$$

$(\mathfrak{A}, \mathsf{L})$ is a *Leibniz quantum compact metric space* if and only if $\left\{a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a\right\} = \mathbb{C}1_{\mathfrak{A}}$.

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Example: Quantum tori

- $G = \mathbb{T}^d$ or some closed subgroup,
- $\mathfrak{A} = C^*(\widehat{G}, \sigma)$ (universal for $U_j U_k = \sigma(j,k) U_{j+k}$),
- α : dual action ($\alpha^z U_j = z^j U_j$).

Some Other Examples

1. unital AF algebras with a faithful tracial state, including Effros-Shen algebras and UHF (Aguilar, L. 15),
2. Hyperbolic group C*-algebras (Rieffel, Ozawa, 05),
3. Nilpotent group C*-algebras (Christ, Rieffel, 16),
4. Connes-Landi spheres (Li, 03)
5. Conformal deformations of quantum metric spaces from spectral triples (L., 15)
6. Group C*-algebras for groups with rapid decay (Antonescu, Christensen, 2004)
7. Curved quantum tori (L., 15)
8. Noncommutative Solenoids (L., Packer, 16)
9. Certain C*-crossed-products (J. Bellissard, M. Marcolli, Reihani, 10), (involves my work on locally compact quantum metric space).

The Lipschitz distance

Definition (Lipchitz distance, L. 16)

The *Lipschitz distance* between two quasi-Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is:

$$\mathrm{LipD}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = \inf\Big\{|\ln(\mathrm{dil}(\varphi)|, |\ln(\mathrm{dil}(\varphi^{-1}))|\Big| \\ \varphi : \mathfrak{A} \to \mathfrak{B} \text{ *-isomorphism}\Big\}$$

where $\mathrm{dil}(\varphi)$ is the norm of φ for the L-seminorms $\mathsf{L}_{\mathfrak{A}}$ and $\mathsf{L}_{\mathfrak{B}}$.

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Theorem (L. 16)

The Lipschitz distance is a *complete extended metric* on the class of quasi-Leibniz quantum compact metric spaces up to full quantum isometry.

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Problem

The Lipschitz distance is ∞ between non *-isomorphic spaces.

The Gromov-Hausdorff Distance

Definition

For any two compact metric spaces (X, m_X) and (Y, m_Y), we define $\mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y)$ as:

$$\left\{ (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \;\middle|\; \begin{array}{l} (Z, \mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\}$$

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Notation

The *Hausdorff distance* on the compact subsets of a metric space (X, m) is denoted by $\mathsf{Haus}_{\mathsf{m}}$.

Definition (Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X, m_X) and (Y, m_Y) is:

$$\inf\left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X), \iota_Y(Y)) : (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \in \mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y) \right\}.$$

A noncommutative Gromov-Hausdorff distance

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How to generalize Gromov's construction to quasi-Leibniz quantum compact metric spaces ?

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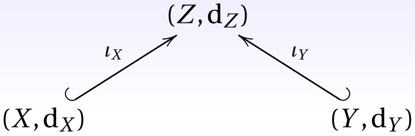

Figure: Isometric Embeddings

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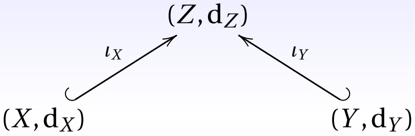

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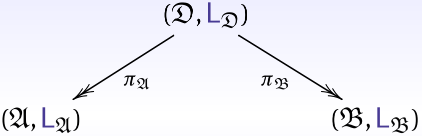

Figure: A tunnel

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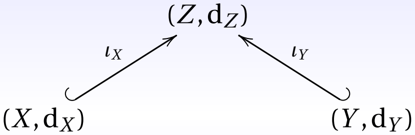

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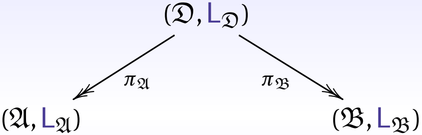

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Choices

- What class $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$ should belong to? *Should we assume a form of quasi-Leibniz inequality?*
- What kind of morphisms $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ should we choose?
- How do we quantify a bridge?

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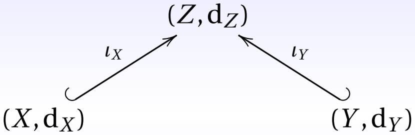

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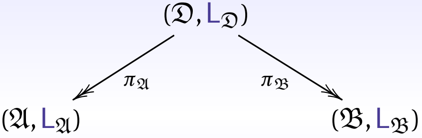

Figure: A tunnel

Previous problems

- *Coincidence problem:* distance zero may not imply *-isomorphism (Rieffel, Wu)
- *Triangle Inequality:* working with quasi-Leibniz $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$ meant abandoning the triangle inequality (Kerr, Rieffel)

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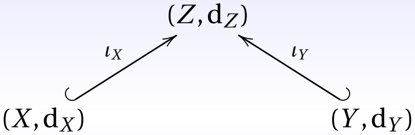

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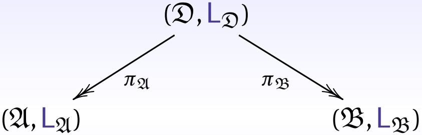

Figure: A tunnel

Previous problem root cause

Noncommutative Gromov-Hausdorff distances construction went out of the category of quasi-Leibniz quantum compact metric spaces. Thus, no easy applications to modules, morphisms, …

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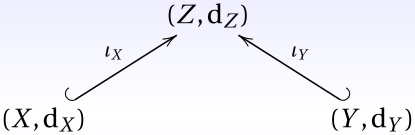

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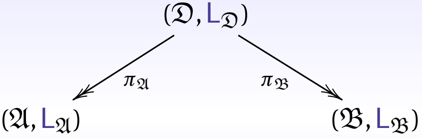

Figure: A tunnel

Solution (L., 13)

- We choose isometric embeddings which are **-morphisms*.
- We restrict embeddings to *F–quasi-Leibniz quantum compact metric space s* or even more specific if desired.

The Dual Gromov-Hausdorff Propinquity

$$(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xleftarrow{\ \pi_{\mathfrak{A}}\ } (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \xrightarrow{\ \pi_{\mathfrak{B}}\ } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$$

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

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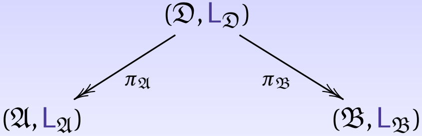

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

Definition (The extent of a tunnel)

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{A}))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}(\mathscr{S}(\mathfrak{B}))\right)\right\}.$$

The Dual Gromov-Hausdorff Propinquity

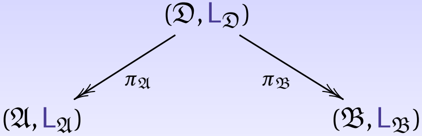

Figure: An F-tunnel: all spaces are F-quasi-Leibniz

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Definition (L. ,13, 14 / special case)

The *dual propinquity* $\Lambda_F^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

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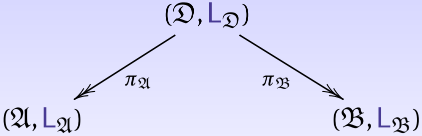

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Definition (L., 13, 14 / special case)

The *dual propinquity* $\Lambda^*_F((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\inf\left\{\chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.$$

Theorem (L., 13)

The dual propinquity is a *complete metric* up to *full quantum isometry*, which induces the same topology on classical compact metric spaces as the Gromov-Hausdorff distance.

Quantum Tori and the quantum propinquity

Endow $\mathbb{T}^d$ with a continuous length function ℓ. Let α be the dual action of $\widehat{\mathbb{Z}_k} \subseteq \mathbb{T}^d$ on $C^*(\mathbb{Z}_k^d, \sigma)$, and set for $a \in \mathfrak{sa}\left(\mathbb{C}^*(\mathbb{Z}_k^d, \sigma)\right)$:

$$\mathsf{L}_\sigma(a) = \sup\left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in \widehat{\mathbb{Z}_k^d} \setminus \{1\} \right\}.$$

L_σ is an L-seminorm (Rieffel, 98).

Theorem (Latrémolière, 2013)

Let $d \in \mathbb{N} \setminus \{0,1\}$, σ a multiplier of $\mathbb{Z}^d$. If for each $n \in \mathbb{N}$, we let $k_n \in \overline{\mathbb{N}}^d$ and σ_n be a multiplier of $\mathbb{Z}_k^d = \mathbb{Z}^d / k_n \mathbb{Z}^d$ such that:

1. $\lim_{n\to\infty} k_n = (\infty, \ldots, \infty)$,
2. the unique lifts of σ_n to $\mathbb{Z}^d$ as multipliers converge pointwise to σ,

then $\lim_{n\to\infty} \Lambda\left(\left(C^*\left(\mathbb{Z}^d, \sigma\right), \mathsf{L}_\sigma\right), \left(C^*\left(\mathbb{Z}_{k_n}^d, \sigma_n\right), \mathsf{L}_{\sigma_n}\right)\right) = 0.$

Finite Dimensional Approximations of quantum tori

Theorem (L. (13))

If for all $n \in \mathbb{N}$, we set $\mathscr{F}_n = C^*(U_n, V_n) = C^*(\mathbb{Z}_n^2, \rho_n)$ where:

$$U_n = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \ldots & \ldots & 1 & 0 \end{pmatrix}, V_n = \begin{pmatrix} 1 & & & & \\ & e^{\frac{2ip_n\pi}{n}} & & & \\ & & e^{\frac{4ip_n\pi}{n}} & & \\ & & & \ddots & \\ & & & & e^{\frac{2i(n-1)p_n\pi}{n}} \end{pmatrix}$$

with $p_n \not\equiv 0 \mod n$, and if $\lim_{n\to\infty} \frac{p_n}{n} = \theta$, then:

$$\lim_{n\to\infty} \Lambda((\mathscr{F}_n, \mathsf{L}_n), (\mathscr{A}_\theta, \mathsf{L}_\theta)) = 0$$

where $\mathscr{A}_\theta = C^*(U,V)$ and U, V are universal unitaries such that $VU = e^{2i\pi\theta}UV$, while L_n and L are L-seminorms from the dual actions, and for some *fixed* continuous length function on $\mathbb{T}^2$.

How to build tunnels ?

Definition (Bridges)

Let $\mathfrak{A}$, $\mathfrak{B}$ be two unital C*-algebras. A *bridge* $\gamma = (\mathfrak{E}, x, \rho_{\mathfrak{A}}, \rho_{\mathfrak{B}})$ from $\mathfrak{A}$ to $\mathfrak{B}$ is given by a unital C*-algebra $\mathfrak{E}$ and two unital *-monomorphisms $\rho_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{E}$ and $\rho_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{E}$, as well as $x \in \mathfrak{E}$ with at least one $\varphi \in \mathscr{S}(\mathfrak{E})$ such that $\varphi = \varphi(x\cdot) = \varphi(\cdot x)$.

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Definition (Length of a bridge)

Let $(\mathfrak{A}_1, \mathsf{L}_1)$ and $(\mathfrak{A}_2, \mathsf{L}_2)$ be two quasi-Leibniz quantum compact metric spaces and $\gamma = (\mathfrak{E}, x, \rho_1, \rho_2)$ be a bridge from $\mathfrak{A}_1$ to $\mathfrak{A}_2$. The *length* $\lambda\left(\gamma \middle| \mathsf{L}_1, \mathsf{L}_2\right)$ of γ is the maximum of:

$$\max_{j \in \{1,2\}} \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_j}} \left(\mathscr{S}(\mathfrak{A}_j), \left\{ \varphi \circ \rho_j : \varphi(x\cdot) = \varphi = \varphi(\cdot x) \in \mathscr{S}(\mathfrak{E}) \right\} \right)$$

and

$$\max_{\{j,k\}=\{1,2\}} \sup_{\substack{a \in \mathfrak{A}_j \\ \mathsf{L}_j(a) \leqslant 1}} \inf_{\substack{b \in \mathfrak{A}_k \\ \mathsf{L}_k(b) \leqslant 1}} \| \rho_j(a) x - x \rho_k(b) \|_{\mathfrak{E}}.$$

Bridges

Theorem (L. (13), Informal)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. For any *bridge* $\gamma = (\mathfrak{E}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from $\mathfrak{A}$ to $\mathfrak{B}$, the following quadruple:

$$(\mathfrak{A} \oplus \mathfrak{B}, (a,b) \mapsto a, (a,b) \mapsto b,$$
$$(a,b) \mapsto \max\left\{ \mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), \frac{1}{\lambda\left(\gamma \middle| \mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right)} \|\pi_{\mathfrak{A}}(a)x - x\pi_{\mathfrak{B}}(b)\|_{\mathfrak{E}} \right\}$$

is a tunnel of extent at most $2\lambda\left(\gamma \middle| \mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right)$.

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We may use the *length* of a bridge to construct a distance on quasi-Leibniz quantum compact metric spaces, the *quantum propinquity* Λ, which dominates the dual propinquity, and induces the same topology on classical compact metric spaces.
All proofs of convergence to date for the dual propinquity are in fact done for the stronger quantum propinquity.

Other examples

1. Conformal perturbations of quantum metrics (L., 15)
2. AF algebras as limits of their inductive sequence in a *metric* sense; UHF and Effros-Shen algebras form continuous families (Aguilar and L., 15),
3. Spheres as limits of full matrix algebras (Rieffel, 15)
4. Nuclear quasi-diagonal quasi-Leibniz quantum compact metric spaces have finite dim approximations (L., 15),
5. There exists an analogue of Gromov's compactness theorem (L., 15)
6. Noncommutative solenoids form a continuous family and have approximations by quantum tori (L. and Packer, 16)
7. Closed balls for the noncommutative Lipschitz distance are totally bounded for Λ (L., 16)

Metrics for Vector Bundles

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$$\langle \omega, \xi \rangle_{C(M)} : x \in M \longmapsto h_x(\omega_x, \xi_x) \in C(M).$$

Thus, ΓV is a $C(M)$-Hilbert module.

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- Let ∇ be a metric connection on ΓV, i.e.:

$$d_X \langle \omega, \xi \rangle = \langle \nabla_X \omega, \xi \rangle + \langle \omega, \nabla_X \xi \rangle.$$

  ∇ defines a norm on a dense subspace of ΓV:

$$\mathsf{D}(\omega) = \max\left\{ \sqrt{\langle \omega, \omega \rangle_{C(M)}}, |||\nabla \omega|||_{\Gamma V}^{\Gamma T M} \right\}.$$

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Our idea is to introduce a metric on objects of the form $(\Gamma V, \langle \cdot, \cdot \rangle_{C(M)}, \mathsf{D}, C(M), \mathsf{L})$.

Metrized quantum vector bundles

Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is given by:

1. $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space,
2. $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ is a left Hilbert module over $\mathfrak{A}$,
3. D is a norm on a dense subspace of $\mathscr{M}$ such that:
    1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
    2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
    3. $\mathsf{D}(a\omega) \leqslant G(\|a\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{D}(\omega))$,
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Full quantum isometries

(θ, Θ) full quantum isometry when θ full quantum isometry between bases and $\Theta(a\xi) = \theta(a)\Theta(\xi)$, Θ linear isomorphism preserving both the norms and the D-norms.

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Example: Classical picture

Hermitian bundles over compact connected Riemannian manifolds.

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Example: Free modules

Given $(\mathfrak{A}, \mathsf{L})$, we set $\langle(a_1,\ldots,a_d),(b_1,\ldots,b_d)\rangle_d = \sum_{j=1}^d a_j b_j^*$ and $\mathsf{L}_d(a_1,\ldots,a_d) = \max\left\{\mathsf{L}(\Re a_j), \mathsf{L}(\Im a_j) : j \in \{1,\ldots,d\}\right\}$. Let $\mathsf{D} = \max\{\|\cdot\|_d, \mathsf{L}_d\}$. Then $(\mathfrak{A}^d, \langle,\rangle_d, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is a metrized quantum vector bundle.

The Heisenberg Modules (Connes, 81; Rieffel)

Fix $\theta \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $d \in \mathbb{N} \setminus \{0\}$ such that $\eth = \theta - \frac{p}{q} \neq 0$.

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1. Start with a representation of $\left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R}^3 \right\}$ on $L^2(\mathbb{R})$:

$$\alpha_{\eth}^{x,y,t}\xi(s) = \exp(i\pi(t + 2xs))\xi(s + \eth y).$$

Promote it to $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.

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2. Let $W_1, W_2 \in U(d)$ with $W_1 W_2 = e^{2i\pi p/q} W_2 W_1$ and $W_1^n = W_2^n = 1$. We get a $\mathscr{A}_\theta = C^*(u_\theta, v_\theta)$-module with:
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$$(u_\theta^n v_\theta^m)\xi = W_1^n W_2^m \alpha_{\eth}^{n,m,0}\xi.$$
3. For Schwarz functions ξ, ω, set:
$$\langle \xi, \omega \rangle_{\mathscr{H}_\theta^{p,q,d}} = \sum_{n,m \in \mathbb{Z}} \langle u_\theta^n v_\theta^m \xi, \omega \rangle_{L^2(\mathbb{R}, \mathbb{C}^d)} u_\theta^n v_\theta^m;$$
complete space of Schwarz functions to the *Heisenberg module* $\mathscr{H}_\theta^{p,q,d}$.

The D-norm

The action of $\mathbb{H}_3$ on Heisenberg modules define a connection. As with the quantum tori, though using a different proof:

Definition (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

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$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

Seminorms from Differential Calculi

- Let α be a nice action of a *Lie group* G on a Banach space $\mathscr{E}$.
- Let $\mathfrak{h}$ a subspace of the *Lie algebra* $\mathfrak{g}$ of G and $\|\cdot\|$ be a norm on $\mathfrak{h}$.

For e in a dense subspace of $\mathscr{E}$, the following limits exist:

$$\nabla e : X \in \mathfrak{h} \mapsto \nabla_X e = X(e) = \lim_{t\to 0} \frac{\alpha^{\exp(tX)} e - e}{t}.$$

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An idea

We could use such differential calculi and associated norms to build quantum metrics.

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Quantum Tori

$G = \mathbb{T}^d$, $\mathscr{E} = \mathscr{A}_\theta$, α is dual action, and $\mathfrak{h} = \mathbb{R}^d$: we get back L_θ for ℓ the path metric length.

Seminorms from Differential Calculi

- Let α be a nice action of a *Lie group* G on a Banach space $\mathscr{E}$.
- Let $\mathfrak{h}$ a subspace of the *Lie algebra* $\mathfrak{g}$ of G and $\|\cdot\|$ be a norm on $\mathfrak{h}$.

For e in a dense subspace of $\mathscr{E}$, the following limits exist:

$$\nabla e : X \in \mathfrak{h} \mapsto \nabla_X e = X(e) = \lim_{t\to 0} \frac{\alpha^{\exp(tX)} e - e}{t}.$$

We denote by $|||\nabla e|||$ the norm of $X \in \mathfrak{h} \mapsto \nabla_X e$. We have (L., 16):

$$\begin{aligned} |||\nabla e||| &= \sup\left\{ \frac{\left\|\alpha^{\exp(X)} e - e\right\|_{\mathscr{E}}}{\|X\|} : X \in \mathfrak{h} \setminus \{0\} \right\} \\ &= \limsup_{\|X\|\to 0} \frac{\left\|\alpha^{\exp(X)} e - e\right\|_{\mathscr{E}}}{\|X\|}. \end{aligned}$$

Heisenberg Modules

$G = \mathbb{H}_3$, $\mathscr{E} = \mathscr{H}_\theta^{p,q,d}$, and $\mathfrak{h} = \mathrm{span}\{P, Q\}$. We get the Yang-Mills connection and our D-norm.

Bridges for modules

Fix $\Omega_{\mathfrak{A}} = (\mathscr{M}_{\mathfrak{A}}, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $\Omega_{\mathfrak{B}} = (\mathscr{M}_{\mathfrak{B}}, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two metrized quantum vector bundles.

Definition (L., 16)

A *modular bridge* $\left(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j)_{j\in J}, (\eta_j)_{j\in J}\right)$ is a bridge $(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ and two families $(\omega_j)_{j\in J} \in \mathscr{M}_{\mathfrak{A}}$, $(\eta_j)_{j\in J} \in \mathscr{M}_{\mathfrak{B}}$ with $\mathsf{D}_{\mathfrak{A}}(\omega_j), \mathsf{D}_{\mathfrak{B}}(\eta_j) \leqslant 1$ for all $j \in J$.

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Definition (L., 16)

The *length* of a modular bridge is the maximum of the length of its basic bridge, and the sum of:

1. the maximum of $\mathsf{Haus}_k(\{\omega_j : j \in J\}, \{\omega : \mathsf{D}_{\mathfrak{A}}(\omega) \leqslant 1)$ and its counterpart in $\Omega_{\mathfrak{B}}$, where:

$$k(\omega,\xi) = \sup\left\{\|\langle\omega,\eta\rangle_{\mathfrak{A}} - \langle\xi,\eta\rangle_{\mathfrak{A}}\|_{\mathfrak{A}} : \mathsf{D}_{\mathfrak{A}}(\eta) \leqslant 1\right\},$$

2. $\max\left\{\|\pi_{\mathfrak{A}}(\langle\omega_j,\omega_k\rangle_{\mathfrak{A}})x - x\pi_{\mathfrak{B}}(\langle\eta_j,\eta_k\rangle_{\mathfrak{B}})\|_{\mathfrak{D}} : j \in J\right\}$.

The modular propinquity

Definition (L., 16)

The *modular propinquity* is the largest pseudo-metric Λ^{mod} such that $\Lambda^{\mathsf{mod}}(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}) \leqslant \lambda(\gamma)$ for any modular γ from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$.

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Theorem (Free modules; L., 16)

If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are quasi-Leibniz quantum compact metric space then:

$$\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant$$
$$\Lambda^{\mathsf{mod}}((\mathfrak{A}^n, \mathsf{D}^n_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}^n, \mathsf{D}^n_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant 2n\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$$

where $\mathsf{D}^n_{\mathfrak{A}}(a_1, \ldots, a_n) = \max_{j=1,\ldots,n}\{\|a_j\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}(\Re(a_j)), \mathsf{L}_{\mathfrak{A}}(\Im(a_j))\}$.

Theorem (L., 16)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and p, q, d fixed. If for all $\theta \in \mathbb{R}$, and $a \in \mathscr{A}_\theta$:

$$\mathsf{L}_\theta(a) = \sup\left\{ \frac{\left\|\beta_\theta^{\exp(ix),\exp(iy)} a - a\right\|_{\mathscr{A}_\theta}}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where β_θ is the dual action, and for all $\xi \in \mathscr{H}_\theta^{p,q,d}$ we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}} \xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where $\eth = \theta - p/q$, then:

$$\lim_{\vartheta \to \theta} \Lambda^{\mathsf{mod}}\left(\left(\mathscr{H}_\vartheta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\vartheta^{p,q,d}}, \mathsf{D}_\vartheta^{p,q,d}, \mathscr{A}_\vartheta, \mathsf{L}_\vartheta\right), \left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)\right) = 0.$$

Thank you!

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