Dirac operator on a noncommutative Toeplitz torus

joint with Fredy Díaz

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Noncommutative Riemannian Geometry

- **Spectral triple:** $(\mathcal{A}, \mathcal{H}, D)$
  $\mathcal{H}$ Hilbert space, $\mathcal{A} \subset \mathrm{B}(\mathcal{H})$ pre C*-algebra, D self-adjoint operator,
  $[D, a] \in \mathcal{B}(\mathcal{H}) \ \ \forall a \in \mathcal{A}, \quad (D + \mathrm{i})^{-1} \in \mathcal{K}(\mathcal{H})$
- **Even spectral triple:** $\gamma = \gamma^* \in \mathrm{B}(\mathcal{H})$ grading operator
  $\gamma^2 = \mathrm{Id}, \quad \gamma D = -D\gamma, \quad \gamma a = a\gamma \ \ \forall a \in \mathcal{A}$
- **Real structure:** J antiunitary operator
  $J^2 = \pm 1, \quad JD = \pm DJ, \quad J\gamma = \pm \gamma J$ (sign dimension dependent)
- **First order condition:** $[a, JbJ^{-1}] = 0 = [[D, a], JbJ^{-1}], \ \ \forall a, b \in \mathcal{A}$
- **Regularity:** $\delta(a) := [|D|, a], \ \ \mathcal{A} \cup [D, \mathcal{A}] \subset \bigcap_{k \in \mathbb{N}} \mathrm{dom}(\delta^k) \subset \mathrm{B}(\mathcal{H})$
- **Finiteness:** $\mathcal{H}^\infty := \bigcap_{k \in \mathbb{N}} \mathrm{dom}(D^k)$ f. g. projective $\mathcal{A}$-module
- **Metric dimension:** $(1 + |D|)^{-n} \in \mathcal{L}^{1^+}(\mathcal{H})$
- **Orientation:** $\exists$ Hochschild cycle s.t. $\sum_k a_k J b_k J^{-1} [D, a_{1k}] \ldots [D, a_{nk}] = \gamma$
- **Poincaré duality:** $K_*(\mathcal{A}) \times K_*(\mathcal{A}) \xrightarrow{\mathrm{ind}} \mathbb{Z}$ non-degenerate

Quantum disc

- **Quantum disc:** Universal C*-algebra generated by

$$Z^*Z - q^2 ZZ^* = 1 - q^2, \qquad q \in (0,1)$$

= C*-algebra generated by Z, $Z^* \in \mathcal{B}(\ell_2(\mathbb{N}))$

$$Z\,e_n = \sqrt{1-q^{2(n+1)}}\,e_{n+1}, \quad Z^*\,e_n = \sqrt{1-q^{2n}}\,e_{n-1}$$

= C*-algebra generated by $S, S^* \in \mathcal{B}(\ell_2(\mathbb{N}))$

$$S\,e_n = e_{n+1}, \quad S^* e_n = e_{n-1}$$

$\cong$ $\mathcal{T}$ (Toeplitz algebra)

- **Remark:** topologically trivial

Toeplitz algebra

- **Notations:**
  - open unit disc $\mathrm{D} := \{z \in \mathbb{C} : |z| < 1\}$ with closure $\bar{\mathrm{D}}$
  - $L_2(\mathrm{D})$ with respect to the Lebesgue measure
  - $A_2(\mathrm{D}) := \{f \in L_2(\mathrm{D}) : \text{analytic in } \mathrm{D}\}$ (Bergman space)
  - $B_{\mathrm{D}} : L_2(\mathrm{D}) \twoheadrightarrow A_2(\mathrm{D}), \quad B_{\mathrm{D}}^2 = B_{\mathrm{D}} = B_{\mathrm{D}}^*$ (Bergman projection)
- **Toeplitz operators:** For all $f \in C(\bar{\mathrm{D}})$,

  $T_f : A_2(\mathrm{D}) \to A_2(\mathrm{D}), \quad T_f(\psi) := B_{\mathrm{D}}(f\psi)$
- **Toeplitz algebra:**

  $\mathcal{T} :=$ C*-alg$\{ T_f : f \in C(\bar{\mathrm{D}}) \} \subset \mathcal{B}(A_2(\mathrm{D}))$
- **Commutators:**

  $[T_f, T_g] \in \mathcal{K}(A_2(\mathrm{D}))$

C*-algebra extension

- **Toeplitz extension:** $A_2(\mathrm{D}) \cong \ell_2(\mathbb{N})$

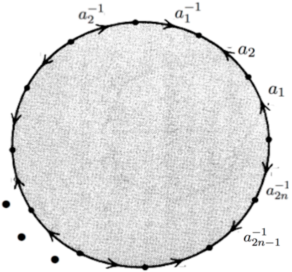
$$0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\ \sigma\ } C(\mathbb{S}^1) \longrightarrow 0$$

- **Symbol map:** $\sigma : \mathcal{T} \longrightarrow C(\mathbb{S}^1)$

$$\sigma(T_f) = f\restriction_{\mathbb{S}^1}, \quad f \in C(\bar{\mathrm{D}})$$

- **Classical picture:**

$$0 \longrightarrow C_0(\mathrm{D}) \longrightarrow C(\bar{\mathrm{D}}) \xrightarrow{\ \restriction_{\mathbb{S}^1}\ } C(\mathbb{S}^1) \longrightarrow 0$$



- **Classical compact surface of genus g:**

$$C(\mathbb{T}_g) \cong \{f \in C(\bar{\mathrm{D}}) : f(a_k(t)) = f(a_k^{-1}(t)),\ t \in [0,1],\ k = 1, \dots, 2g\}$$

- **Classical compact surface of genus g:** $\sigma : \mathcal{T} \longrightarrow C(\mathbb{S}^1)$

$$C(\mathbb{T}_{q,g}) := \{f \in \mathcal{T} : \sigma(f)(a_k(t)) = \sigma(f)(a_k^{-1}(t)),\ t \in [0,1],\ k = 1, \dots, 2g\}$$

- **Noncommutative Toeplitz torus:**

$$C(\mathbb{T}_q) = \{f \in \mathcal{T} : \sigma(f)(\mathrm{e}^{\mathrm{i}t}) = \sigma(f)(-\mathrm{i}\mathrm{e}^{-\mathrm{i}t}),\ \sigma(f)(\mathrm{e}^{-\mathrm{i}t}) = \sigma(f)(\mathrm{i}\mathrm{e}^{\mathrm{i}t}),\ t \in [0, \tfrac{\pi}{2}]\}$$

K-Theory of classical oriented surfaces

- **C*-algebra extension:** $C(\mathbb{T}_g) \subset C(\bar{\mathrm{D}})$

$$0 \longrightarrow C_0(\mathrm{D}) \longrightarrow C(\mathbb{T}_g) \xrightarrow{\restriction_{\mathbb{S}^1}} C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1) \longrightarrow 0$$

- **Six-term exact sequence:**

$$\begin{array}{ccccc} K_0(C_0(\mathrm{D})) & \longrightarrow & K_0(C(\mathbb{T}_g)) & \longrightarrow & K_0(C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1)) \\ \uparrow \text{ind} & & & & \downarrow \text{exp} \\ K_1(C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1)) & \longleftarrow & K_1(C(\mathbb{T}_g)) & \longleftarrow & K_1(C_0(\mathrm{D})) \end{array}$$

- **K-groups:**

$$K_*(C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1)) = K_*(C_0((0,1)) \oplus \ldots \oplus C_0((0,1)) \oplus \mathbb{C}\mathbf{1})$$

$$\Rightarrow \quad K_0(C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1)) = \mathbb{Z}, \qquad K_1(C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1)) = \mathbb{Z}^{2g}$$

$$K_*(C_0(\mathrm{D})) = K_*(C_0(\mathbb{R}^2)) = K_*(\Sigma^2\mathbb{C}) = K_*(\mathbb{C})$$

$$\Rightarrow \quad K_0(C_0(\mathrm{D})) = \mathbb{Z}, \qquad K_1(C_0(\mathrm{D})) = 0.$$

K-Theory of classical oriented surfaces

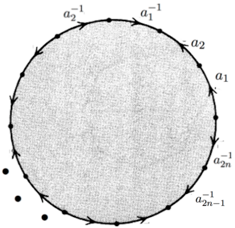
- **The index map:** $z \in C(\bar{\mathrm{D}}),\ \ z(r\mathrm{e}^{\mathrm{i}t}) := r\mathrm{e}^{\mathrm{i}t},\ \ z = |z|u,\ \ u \in C(\mathbb{S}^1)$

$$\begin{array}{ccccc}
\mathbb{Z} \cong K_0(C_0(\mathrm{D})) & \longrightarrow & K_0(C(\mathbb{T}_g)) & \xrightarrow{[1]\mapsto[1]} & \mathbb{Z} \\
\uparrow{\scriptstyle \mathrm{ind}} & & & & \downarrow{\scriptstyle \mathbf{0}} \\
\mathbb{Z}^{2g} \cong K_1(C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1)) & \longleftarrow & K_1(C(\mathbb{T}_g)) & \longleftarrow & 0
\end{array}$$

$$\mathbb{Z} \ni k \ \stackrel{\cong}{\longmapsto}\ \left[\begin{pmatrix} |z|^2 & \sqrt{1-|z|^2}\,|z|u^{*k} \\ u^k|z|\sqrt{1-|z|^2} & 1-|z|^2 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in K_0(C_0(\mathrm{D}))$$

$[v] \in K_1(C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1))$ w.l.o.g. $v \in C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1) \subset C(\mathbb{S}^1) = C(\partial\bar{\mathrm{D}})$

$$\mathrm{ind}[v] = \left[\begin{pmatrix} |z|^2 & \sqrt{1-|z|^2}\,|z|v^{*} \\ v|z|\sqrt{1-|z|^2} & 1-|z|^2 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \longmapsto \mathrm{wind}(v) \in \mathbb{Z}$$



$$K_1(C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1)) \ni [v] \longmapsto \mathrm{ind}[v] = \mathrm{wind}(v) = 0 \in \mathbb{Z} \cong K_0(C_0(D))$$

$$\begin{array}{ccccc} \mathbb{Z} & \hookrightarrow & K_0(C(\mathbb{T}_g)) & \xrightarrow{[1]\mapsto[1]} & \mathbb{Z} \\ {\scriptstyle 0}\uparrow & & & & \downarrow{\scriptstyle \mathbf{0}} \\ \mathbb{Z}^{2g} & \xleftarrow{\cong} & K_1(C(\mathbb{T}_g)) & \longleftarrow & 0 \end{array}$$

$$\Rightarrow \quad K_0(C(\mathbb{T}_g)) \cong \mathbb{Z}^2, \qquad K_1(C(\mathbb{T}_g)) \cong \mathbb{Z}^{2g}$$

K-Theory of compact quantum surfaces

- **C*-algebra extension:**

$$0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \longrightarrow C(\mathbb{T}_{q,g}) \xrightarrow{\sigma} C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1) \longrightarrow 0$$

- **Six-term exact sequence:**

$$\begin{array}{ccccc} K_0(\mathcal{K}(\ell_2(\mathbb{N}))) & \longrightarrow & K_0(C(\mathbb{T}_{q,g})) & \longrightarrow & K_0(C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1)) \\ \uparrow \text{ind} & & & & \downarrow \exp \\ K_1(C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1)) & \longleftarrow & K_1(C(\mathbb{T}_{q,g})) & \longleftarrow & K_1(\mathcal{K}(\ell_2(\mathbb{N}))) \end{array}$$

$$K_*(\mathcal{K}(\ell_2(\mathbb{N}))) = K_*(\mathcal{K}(\ell_2(\mathbb{N})) \otimes \mathbb{C}) = K_*(\mathbb{C}) = K_*(\Sigma^2\mathbb{C}) = K_*(C_0(\mathrm{D}))$$

- **Index map:**

$$\mathrm{ind} : K_0(C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1)) \subset \frac{\mathcal{B}(\ell_2(\mathbb{N}))}{\mathcal{K}(\ell_2(\mathbb{N}))} = \mathcal{C}(\ell_2(\mathbb{N})) \xrightarrow{\mathrm{ind}} \mathbb{Z} \cong K_0(\mathcal{K}(\ell_2(\mathbb{N})))$$

$$K_0(C(\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1)) \ni [v] \mapsto T_{|z|v} \in C(\mathbb{T}_{q,g}) \mapsto \mathrm{ind}(T_{|z|v}) = \mathrm{wind}(v) = 0 \in \mathbb{Z}$$

K-Theory of compact quantum surfaces

- **Generalized Bott projections:**

$$\mathrm{ind}[v] = \left[\begin{pmatrix} |T^*_{|z|v}|^2 & T_{|z|v}\sqrt{1-|T_{|z|v}|^2} \\ \sqrt{1-|T_{|z|v}|^2}\, T^*_{|z|v} & 1-|T_{|z|v}|^2 \end{pmatrix} \right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0(\mathcal{K}(\ell_2(\mathbb{N})))$$

$$\mathrm{ind}[v] = \mathrm{ind}(T_{|z|v}) = \mathrm{wind}(v) = 0$$

- **Six-term exact sequence:**

$$\begin{array}{ccccc} \mathbb{Z} & \hookrightarrow & K_0(C(\mathbb{T}_{q,g})) & \xrightarrow{[1]\mapsto[1]} & \mathbb{Z} \\ {\scriptstyle 0}\uparrow & & & & \downarrow{\scriptstyle \mathbf{0}} \\ \mathbb{Z}^{2g} & \xleftarrow{\cong} & K_1(C(\mathbb{T}_{q,g})) & \longleftarrow & 0 \end{array}$$

$$\Rightarrow \quad K_0(C(\mathbb{T}_{q,g})) \cong \mathbb{Z}^2 \cong K_0(C(\mathbb{T}_g)), \quad K_1(C(\mathbb{T}_{q,g})) \cong \mathbb{Z}^{2g} \cong K_1(C(\mathbb{T}_g))$$

Classical spectral triple on the flat torus

- **Spectral tripel** $(\mathcal{A}, \mathcal{H}, D)$**:**
  - $\mathcal{H} = L_2([0,1]\times[0,1]) \oplus L_2([0,1]\times[0,1])$
  - $\mathcal{A} := C^\infty(\mathbb{R}^2/\mathbb{Z}^2)$
  - $D = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \left(\frac{\partial}{\partial z}\right)^* & 0 \end{pmatrix}$ on $\mathrm{dom}(D) = \mathcal{H}_0 \oplus \mathcal{H}_0,$

    $\mathcal{H}_0 := \{f \in AC(\mathbb{R}^2/\mathbb{Z}^2) : \frac{\partial f}{\partial z} \in L_2([0,1]\times[0,1])\}$

- **Bounded perturbation:** $D_0 \in \mathcal{B}(\mathcal{H})$

  $(\mathcal{A}, \mathcal{H}, D)$ spectral triple $\Longrightarrow$ $(\mathcal{A}, \mathcal{H}, D + D_0)$ spectral triple

- **Conformal mapping:**

  $[0,1]\times[0,1] \xrightarrow{\cong} \bar{\mathbb{D}} \Longrightarrow D \longmapsto h\,D\,h, \quad h > 0 \ \ (h \in \mathcal{Z}(\mathcal{A})$ or $\mathcal{A}^{\mathrm{op}})$

Dirac operator on the quantum disc

- **Cauchy-Riemann operators:**

$$\frac{\partial}{\partial z} : \mathrm{dom}(\tfrac{\partial}{\partial z}) \subset A_2(\mathrm{D}) \longrightarrow A_2(\mathrm{D}), \quad \psi \longmapsto \psi' = \frac{\partial \psi}{\partial z}$$

- **Orthonormal basis:**

$$e_k := \sqrt{\tfrac{k+1}{\pi}}\, z^k, \ k \in \mathbb{N} \implies \frac{\partial}{\partial z} e_k = \sqrt{k(k+1)}\, e_{k-1}$$

- **Bounded perturbation:**

$$\partial_z e_k = k\, e_{k-1} \implies \partial_z - \frac{\partial}{\partial z} \in \mathcal{B}(A_2(\mathrm{D}))$$

- **Quantum Disc :** $\mathcal{A} :=$ *-alg$\{Z, Z^*\} \subset \mathcal{B}(A_2(\mathrm{D}))$

$$Z e_k = \lambda_{k+1}\, e_{k+1}, \quad Z^* e_k = \lambda_k\, e_{k-1}, \quad \lambda_k := \sqrt{1 - q^{2k}}$$

$\Rightarrow$ $[\partial_z, Z] e_k \sim \lambda_k\, e_k, \quad [\partial_z, Z^*] e_k \sim -\lambda_k\, e_{k-2}$

$$D := \begin{pmatrix} 0 & \partial_z \\ \partial_z^* & 0 \end{pmatrix} \implies [D, a] \in \mathcal{B}(A_2(\mathrm{D})), \quad (D + \mathrm{i})^{-1} \in \mathcal{K}(A_2(\mathrm{D}))$$

$\Rightarrow$ $(\mathcal{A}, A_2(\mathrm{D}), D)$ is a spectral triple for $\mathcal{A} \subset \mathcal{T} = \overline{\mathcal{A}}$

Alternative description of the Toeplitz torus

- **Alternative description of the Toeplitz algebra:**

- $L_2(\mathbb{S}^1) \cong \ell_2(\mathbb{Z}) = \overline{\mathrm{span}}\{e_k := \frac{1}{\sqrt{2\pi}} u^k : k \in \mathbb{Z}\}, \quad u(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}t}, \quad u \in C(\mathbb{S}^1)$
- $P : L_2(\mathbb{S}^1) \twoheadrightarrow \overline{\mathrm{span}}\{u^k : k \geq 0\} \cong \ell_2(\mathbb{N}), \quad P^2 = P = P^*$
- $T_f : \ell_2(\mathbb{N}) \longrightarrow \ell_2(\mathbb{N}), \quad T_f(\psi) := P(f\psi), \quad f \in C(\mathbb{S}^1), \quad \psi \in L_2(\mathbb{S}^1)$
- $\mathcal{T} :=$ C*-alg$\{\, T_f : f \in C(\mathbb{S}^1)\,\} \subset \mathcal{B}(\ell_2(\mathbb{N}))$
- $0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(\mathbb{S}^1) \longrightarrow 0, \quad \sigma(T_f) = f$

- **Noncommutative Toeplitz torus:**

$$C(\mathbb{S}^1 \wedge \mathbb{S}^1) = \{f \in C(\mathbb{S}^1) : f(\mathrm{e}^{\mathrm{i}t}) = f(-\mathrm{i}\mathrm{e}^{-\mathrm{i}t}),\ f(\mathrm{e}^{-\mathrm{i}t}) = f(\mathrm{i}\mathrm{e}^{\mathrm{i}t}),\ t \in [0, \tfrac{\pi}{2}]\}$$

$$C(\mathbb{T}_q) = \text{C*-alg}\{\, T_f \in \mathcal{T} : f \in C(\mathbb{S}^1 \wedge \mathbb{S}^1) \subset C(\mathbb{S}^1)\,\}$$

- **Dimension drop:** $\dim(\mathrm{D}) = 2, \quad \dim(\mathbb{S}^1) = 1$

- **Hilbert space representation:** $u(t) = \mathrm{e}^{\mathrm{i}t}, \quad e_m(t) = \frac{1}{\sqrt{2\pi}} u^m(t)$

$$f = \sum_{k\in\mathbb{Z}} \hat{f}(j)\, u^j \in C^{(1)}(\mathbb{S}^1) \subset L_2(\mathbb{S}^1) \quad \text{(Fourier series)}$$

$$\Rightarrow \quad T_f(e_m) = P\Big(\sum_{k\in\mathbb{Z}} \hat{f}(k)\, e_{k+m}\Big) = \sum_{k\in\mathbb{N}} \hat{f}(k-m)\, e_k$$

- **Derivations:**

$$\partial_z e_k = k\, e_{k-1} \quad \Longrightarrow \quad \partial_z = -\mathrm{i}\,\bar{u}\,\frac{\mathrm{d}}{\mathrm{d}t}$$

- **Bounded commutators:** $\forall f \in C^{(1)}(\mathbb{S}^1)$

$$\Rightarrow \quad [\partial_z, T_f](e_m) = \sum_{k\in\mathbb{N}} k\, \hat{f}(k-m)\, e_{k-1} - m\, \hat{f}(k-m+1)\, e_k$$

$$= \sum_{k\in\mathbb{N}} (k-m+1)\, \hat{f}(k-m+1)\, e_k = -\mathrm{i} T_{\bar{u} f'}(e_m)$$

$$\Rightarrow \quad [\partial_z, T_f] = -\mathrm{i} T_{\bar{u} f'} \in \mathrm{B}(\ell_2(\mathbb{N})), \quad [\partial_z^*, T_f] = -([\partial_z, T_{\bar{f}}])^* \in \mathrm{B}(\ell_2(\mathbb{N}))$$

Spectral triple on the noncommutative Toeplitz torus

- **Hilbert space:** $\mathcal{H} := \ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N})$
- **Algebra:** $\mathcal{A} :=$ *-alg$\{T_f \oplus T_f : f \in C(\mathbb{S}^1 \wedge \mathbb{S}^1) \cap C^{(1)}(\mathbb{S}^1)\}$

  $\Longrightarrow \ \overline{\mathcal{A}} \cong C(\mathbb{T}_q)$
- **Dirac operator:** $D := \begin{pmatrix} 0 & \partial_z \\ \partial_z^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i}\,\bar{u}\,\frac{\mathrm{d}}{\mathrm{d}t} \\ -\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,u & 0 \end{pmatrix}$
- **Eigenfunctions:** $\mathcal{H} = \overline{\mathrm{span}}\{b_k : k \in \mathbb{Z}\}$

  $b_k := e_{k-1} \oplus e_k, \quad b_{-k} := -e_{k-1} \oplus e_k, \quad k > 0, \quad b_0 = 0 \oplus e_0$

$\Rightarrow \ D(b_k) = k\,b_k, \quad k \in \mathbb{Z} \ \Longrightarrow \ \mathrm{spec}(D) = \mathbb{Z}$

$\Rightarrow \ (D+\mathrm{i})^{-1} \in \mathcal{K}(\mathcal{H}),\ (|D|+1)^{-1} \in \mathcal{L}^{1^+}(\mathcal{H}) \ \Longrightarrow$ metric dimension $= 1$

- **Fredholm module:** $Ne_k := k\,e_k$ (number operator), $Se_k = e_{k+1}$ (shift)

$\Rightarrow \ D = F\,|D| = \begin{pmatrix} 0 & \bar{u} \\ u & 0 \end{pmatrix} \begin{pmatrix} N+1 & 0 \\ 0 & N \end{pmatrix} = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} \begin{pmatrix} N+1 & 0 \\ 0 & N \end{pmatrix}$

- **Fredholm index:** $\mathrm{ind}(D) = \mathrm{ind}(S^*) = 1 \neq 0$

Grading and real structure

- **Grading:** $\gamma := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ on $\mathcal{H} := \ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N})$

$\Rightarrow$ $\gamma^2 = 1, \quad \gamma D = -D\gamma, \quad \gamma a = a\gamma$ for all $a \in \mathcal{A}$

$\Rightarrow$ $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is an even spectral triple ($\Rightarrow$ "even dimensional")

- **Real Structure:** $J \in \mathrm{B}(\mathcal{H})$ antiunitary operator s. t.

  $J^2 = \pm 1, \quad JD = \pm DJ, \quad J\gamma = \pm\gamma J$ (sign dimension dependent)

  $\dim(\mathbb{T}_q) = 2 \implies J^2 = -1, \quad JD = DJ, \quad J\gamma = -\gamma J$

- **Nonexistence:** $\mathcal{H} = \overline{\mathrm{span}}\{b_k : k \in \mathbb{Z}\}, \quad D\, b_k = k\, b_k$

  $JD = DJ \implies Jb_k = \alpha_k b_k, \quad |\alpha_k| = 1$ (antiunitary)

$\Rightarrow$ $J^2 b_k = J(\alpha_k b_k) = \bar{\alpha}_k\, Jb_k = \bar{\alpha}_k \alpha_k b_k = b_k$ (antiunitary) $\Rightarrow$ $J^2 = 1$

- **Problem:** multiplicity of eigenvalues $= 1$

$\Rightarrow$ metric dimension $= 1$

First order condition

- **First order condition:** $J \in \mathrm{B}(\mathcal{H})$ antiunitary operator such that

  $$[a, JbJ^{-1}] = 0, \quad [[D, a], JbJ^{-1}] = 0$$

$\Rightarrow$ $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$-action on $\mathcal{H}$

- **Nonexistence:** $\mathcal{H} = \ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N})$

$$\Rightarrow \mathrm{B}(\mathcal{H}) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \mathrm{B}(\ell_2(\mathbb{N})) \right\}$$

$$\left\{ \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} : k \in \mathcal{K}(\ell_2(\mathbb{N})) \right\} \subset \mathcal{A}$$

$$\Rightarrow \left[\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right] = \begin{pmatrix} [k, a_{11}] & [k, a_{12}] \\ [k, a_{21}] & [k, a_{22}] \end{pmatrix} \stackrel{!}{=} 0 \quad \forall k \in \mathcal{K}(\ell_2(\mathbb{N}))$$

$$\Leftrightarrow a_{ij} \in \mathcal{K}(\ell_2(\mathbb{N}))' = \mathbb{C}$$

$$\Rightarrow \dim(\mathcal{A}') = 4 < \infty = \dim(J\mathcal{A}J^{-1})$$

Odd spectral triple and real structure

- **Real Structure for odd spectral triple:** $(\mathcal{A}, \mathcal{H}, D)$

  metric dimension $= 1 \quad \Longrightarrow \quad J^2 = 1, \quad JD = -DJ$

- **Eigenvectors:** $\mathcal{H} = \overline{\text{span}}\{b_k : k \in \mathbb{Z}\}, \quad D\, b_k = k\, b_k$

$\Rightarrow$ $\text{spec}(D) = -\text{spec}(D)$

- **Antiunitary operator:** $J b_k := b_{-k} \quad \Longrightarrow \quad J^2 = 1$

$\Rightarrow$ $DJb_k = -kb_{-k} = -JDb_k \quad \Longrightarrow \quad JD = -DJ$

- **First order condition:** As a real linear operator: $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ $JT_f J^{-1} = JT_f J = T_{\hat{f}}, \quad \forall f \in C(\mathbb{S}^1)$

$\widehat{C}(\mathbb{S}^1) := C(\mathbb{S}^1)$ with scalar multiplication $\alpha \cdot \hat{f} = \widehat{\bar{\alpha} f}$

$\Rightarrow$ $[T_g, JT_f J] = [T_g, T_{\hat{f}}], \quad [[\partial_z, T_g], JT_f J] = -\mathrm{i}[T_{\bar{u}g'}, T_{\hat{f}}] \in \mathcal{K}(\ell_2(\mathbb{N}))$

$\Rightarrow$ $[a, JbJ^{-1}] \in \mathcal{K}(\mathcal{H}), \; [[D, a], JbJ^{-1}] \in \mathcal{K}(\mathcal{H})$ for all $a, b \in \mathcal{A}$

$\Rightarrow$ first order condition satisfied "up to compacts"

Regularity

- **Regularity:** $\delta(a) := [|D|, a]$

  $\mathcal{T}^\infty :=$ *-alg$\{T_f \oplus T_f : f \in C^\infty(\mathbb{S}^1)\}$

  $C^\infty(\mathbb{S}^1 \wedge \mathbb{S}^1) := C(\mathbb{S}^1 \wedge \mathbb{S}^1) \cap C^\infty(\mathbb{S}^1)$

  $\mathcal{A}^\infty :=$ *-alg$\{T_f \oplus T_f : f \in C^\infty(\mathbb{S}^1 \wedge \mathbb{S}^1)\}$

  $f \in C^\infty(\mathbb{S}^1 \wedge \mathbb{S}^1)$

$\Rightarrow$ $[\partial_z, T_f] = -\mathrm{i} T_{\bar{u} f'}$,

$[|\partial_z|, T_f](e_m) = [N, T_f](e_m) = \sum_{k \in \mathbb{N}} (k-m) f(k-m) e_k = -\mathrm{i} T_{f'}(e_m)$

$\Rightarrow$ $[|D|, T_f \oplus T_f] = -\mathrm{i}(T_{f'} \oplus T_{f'}) \in \mathcal{A}^\infty \subset \mathcal{T}^\infty$,

$[D, T_f \oplus T_f] = \mathrm{i}(T_{\bar{u} f'} \oplus T_{u f'}) \in \mathcal{T}^\infty$

$\Rightarrow$ $\mathcal{A}^\infty \cup [D, \mathcal{A}^\infty] \subset \mathcal{T}^\infty \subset \bigcap_{k \in \mathbb{N}} \mathrm{dom}(\delta^k) \subset \mathrm{B}(\mathcal{H})$

Finiteness

- **Finiteness:** $P : L_2(\mathbb{S}^1) \longrightarrow \ell_2(\mathbb{N})$ Toeplitz projection

  $\mathcal{D}^\infty := P\big(C^\infty(\mathbb{S}^1 \wedge \mathbb{S}^1)\big) \subset P\big(L_2(\mathbb{S}^1)\big) = \ell_2(\mathbb{N})$ dense

$\Rightarrow$ $\mathcal{D}^\infty \oplus \mathcal{D}^\infty \subset \mathcal{H}$ core of analytic vectors for D

$(1-SS^*) = (1-T_u T_{\bar{u}}) \in \mathcal{K}(\ell_2(\mathbb{N})) \subset C(\mathbb{T}_q)$ projection onto $\mathbb{C}e_0$

$C^\infty(\mathbb{T}_q) :=$ *-alg$\{T_f : f \in C^\infty(\mathbb{S}^1 \wedge \mathbb{S}^1)\}$

$\mathcal{H}^\infty := C^\infty(\mathbb{T}_q)(1-SS^*) \subset C(\mathbb{T}_q)(1-T_u T_{\bar{u}})$ f. g. projective module

$\langle a, b\rangle_{\mathcal{H}^\infty} := \mathrm{Tr}_{\ell_2(\mathbb{N})}(a^* b) \quad a, b \in \mathcal{H}^\infty$

$f = \sum_{k\in\mathbb{Z}} \hat{f}(k) u^k \in C^\infty(\mathbb{S}^1 \wedge \mathbb{S}^1) \subset L_2(\mathbb{S}^1) \Rightarrow Pf = \sum_{k\in\mathbb{N}} \hat{f}(k) e_k \in \ell_2(\mathbb{N})$

$\Rightarrow$ $\langle T_f, T_f\rangle_{\mathcal{H}^\infty} = \sum_{k\in\mathbb{N}} |\hat{f}(k)|^2 = \langle Pf, Pf\rangle \quad (S^{*k}S^k = 1,\ S^k S^{*k}(1-SS^*) = 0)$

$\Rightarrow$ $\iota : \mathcal{D}^\infty \hookrightarrow \mathcal{H}^\infty,\ \ Pf \mapsto T_f(1-SS^*), \quad f \in C^\infty(\mathbb{S}^1 \wedge \mathbb{S}^1)$ isometry

$\Rightarrow$ $\iota(\mathcal{D}^\infty) \oplus \iota(\mathcal{D}^\infty) \subset \mathcal{H}^\infty \oplus \mathcal{H}^\infty \subset \bigcap_{m\in\mathbb{N}} \mathrm{dom}(D^m)$

$\Rightarrow$ $\mathcal{H}^\infty \oplus \mathcal{H}^\infty = C^\infty(\mathbb{T}_q)(1-SS^*) \oplus C^\infty(\mathbb{T}_q)(1-SS^*)$ core for D^m

Failure of Poincaré duality

- **Spectral triple:** $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$, $\mathcal{H} = \ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N})$
- **Opposite Representation:** $a^{\mathrm{op}} := J\, a^* J^{-1}$, $a \in \mathcal{A} = C(\mathbb{T}_q)$
- $\boldsymbol{K_0}$**-group:** $K_0(C(\mathbb{T}_q)) = K_0(\mathcal{K}(\ell_2(\mathbb{N}))) \oplus K_0(\mathbb{C})$
- **Poincaré duality:** $K_0(C(\mathbb{T}_q)) \times K_0(C(\mathbb{T}_q)) \xrightarrow{\mathrm{ind}} \mathbb{Z}$ non-deg.

- Generators: $1 \in C(\mathbb{T}_{q,n})$, $\quad p_1, q_1 \in C(\mathbb{T}_{q,n})$ 1-dimensional projection
- $K_0(C(\mathbb{T}_{q,n})) = \mathbb{Z}[p_1] \oplus \mathbb{Z}[1] = \mathbb{Z}[q_1] \oplus \mathbb{Z}[1]$
- $p_1\, q_1^{\mathrm{op}} = q_1^{\mathrm{op}}\, p_1 \implies p_1\, q_1^{\mathrm{op}} = p_1$ or $p_1\, q_1^{\mathrm{op}} = 0$ Finite dimensional!
- $\langle [p_1], [p_1] \rangle = \langle [p_1], [q_1] \rangle = \mathrm{ind}(p_1\, q_1^{\mathrm{op}}\, S^*\, p_1\, q_1^{\mathrm{op}}) = 0$
- $\langle [p_1], [1] \rangle = \mathrm{ind}(p_1\, S^*\, p_1) = 0 = \mathrm{ind}(p_1^{\mathrm{op}}\, S^*\, p_1^{\mathrm{op}}) = \langle [1], [p_1] \rangle$
- $\langle [1], [1] \rangle = \mathrm{ind}(S^*) = 1$

$\Rightarrow$ $\langle k[p_1] + m[1], l[p_1] + n[1] \rangle = m\, n$ Degenerated!

Failure of Orientation

- **Orientation:** $\sum_k a_k J b_k^* J^{-1} \otimes a_{1,k} \otimes \ldots \otimes a_{n,k}$ Hochschild cycle s.t. $\sum_k a_k J b_k J^{-1} [D, a_{1,k}] \ldots [D, a_{n,k}] = \gamma, \qquad a_k, b_k, a_{1,k}, \ldots, a_{n,k} \in \mathcal{A}$
- **Metric dimension = 1:**

  $\sum_k a_k J b_k J^{-1} [D, a_{1,k}]$ odd, γ even $\Rightarrow$ Contradiction!
- **Dimension = 2:** $J T_f J^{-1} = T_{\hat{f}} \;\Rightarrow\; a_{0,k} := a_k J b_k J^{-1} \in \mathcal{A}$

$$\sum_k a_{0,k} [D, a_{1,k}][D, a_{2,k}] = \begin{pmatrix} \sum_k a_{0,k} [\partial_z, a_{1,k}][\partial_z^*, a_{2,k}] & 0 \\ 0 & \sum_k a_{0,k} [\partial_z^*, a_{1,k}][\partial_z, a_{2,k}] \end{pmatrix}$$

$$\sigma([\partial_z, T_f]) = -\mathrm{i}\bar{u} f', \quad \sigma([\partial_z^*, T_f]) = -\mathrm{i} u f'$$

$$\Rightarrow \sigma\Big(\sum_k a_{0,k} [\partial_z, a_{1,k}][\partial_z^*, a_{2,k}]\Big) = \sigma\Big(\sum_k a_{0,k} [\partial_z^*, a_{1,k}][\partial_z, a_{2,k}]\Big)$$

$$\Rightarrow \sigma\Big(\sum_k a_{0,k} [D, a_{1,k}][D, a_{2,k}]\Big) \neq \sigma(\gamma) = \gamma$$