RADBOUD UNIVERSITEIT NIJMEGEN

THERE AND BACK AGAIN: FROM THE BORSUK-ULAM THEOREM TO QUANTUM SPACES

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Using the Borsuk-Ulam Theorem

Lectures on Topological Methods in Combinatorics and Geometry

The Borsuk-Ulam Theorem

Theorem (Borsuk-Ulam)

Let n be a positive natural number. If $f\colon S^n \to \mathbb{R}^n$ is continuous, then there exists a pair $(p, -p)$ of antipodal points on S^n such that $f(p) = f(-p)$.

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The logical negation of the theorem

There exists a continuous map $f\colon S^n \to \mathbb{R}^n$ such that for all pairs $(p, -p)$ of antipodal points on S^n we have $f(p) \neq f(-p)$.

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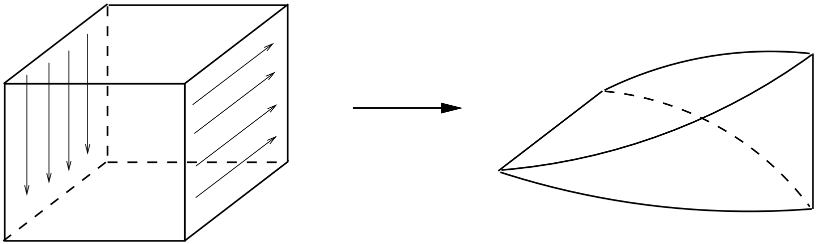
There exists a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^n \to S^{n-1}$.

Theorem (equivariant formulation)

Let n be a positive natural number. There does not exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^n \to S^{n-1}$.

Equivariant join construction

For any topological spaces X and Y, one defines the join space $X * Y$ as the quotient of $[0,1] \times X \times Y$ by a certain equivalence relation:



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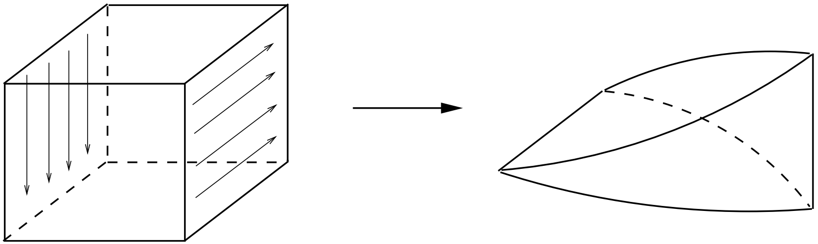

If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G, then the diagonal action of G on the join $X * G$ is again continuous and free.

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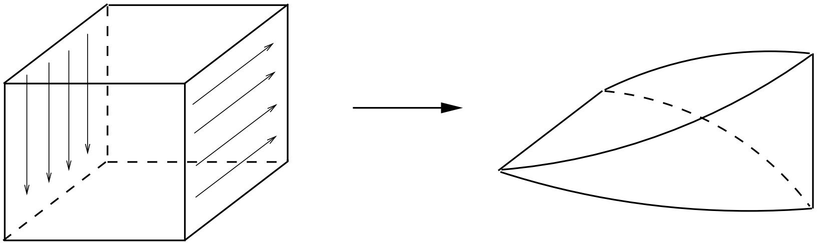

If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G, then the diagonal action of G on the join $X * G$ is again continuous and free. In particular, for the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on S^{n-1}, we obtain a $\mathbb{Z}/2\mathbb{Z}$-equivariant identification $S^n \cong S^{n-1} * \mathbb{Z}/2\mathbb{Z}$ for the antipodal and diagonal actions respectively.

Join formulation and classical generalization

Thus the Borsuk-Ulam Theorem is equivalent to:

Theorem (join formulation)

Let n be a positive natural number. There does not exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^{n-1} * \mathbb{Z}/2\mathbb{Z} \to S^{n-1}$.

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This naturally leads to:

A classical Borsuk-Ulam-type conjecture

Let X be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group G. Then, for the diagonal action of G on $X * G$, there does not exist a G-equivariant continuous map $f : X * G \to X$.

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At the moment, the conjecture is known to hold under the assumption of local triviality.

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Theorem (Gelfand-Naimark I)

Every commutative C-algebra is naturally isomorphic to the algebra of all continuous complex-valued vanishing-at-infinity functions on a locally compact Hausdorff space.*

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Copernican-style revolution

Given a compact Hausdorff space of points, we can define the C*-algebra of functions on the space, but the central concept is that of a commutative C*-algebras, and points appear as characters (algebra homomorphisms into $\mathbb{C}$) rather than as primary objects. We think of noncommutative unital C*-algebras as algebras of functions on compact quantum spaces.

What is a compact quantum group?

Definition (S. L. Woronowicz)

A compact quantum group is a unital C^*-algebra H with a given unital $*$-homorphism $\Delta\colon H \longrightarrow H \otimes_{\min} H$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\quad\Delta\quad} & H \underset{\min}{\otimes} H \\ \Big\downarrow{\scriptstyle\Delta} & & \Big\downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ H \underset{\min}{\otimes} H & \xrightarrow[\mathrm{id}\otimes\Delta]{} & H \underset{\min}{\otimes} H \underset{\min}{\otimes} H \end{array}$$

commutes and the two-sided cancellation property holds:

$$\{(a\otimes 1)\Delta(b) \mid a, b \in H\}^{\mathrm{cls}} = H \underset{\min}{\otimes} H = \{\Delta(a)(1\otimes b) \mid a, b \in H\}^{\mathrm{cls}}.$$

Here “cls” stands for “closed linear span”.

Free actions of compact quantum groups

Let A be a unital C^*-algebra and $\delta : A \to A \otimes_{\min} H$ a unital $*$-homomorphism. We call δ a coaction of H on A (or an action of the compact quantum group (H, Δ) on A) iff

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\ h \in H\}^{\mathrm{cls}} = A \otimes_{\min} H$ (counitality)
3. $\ker \delta = 0$ (injectivity).

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Definition (D. A. Ellwood)

A coaction δ is called free iff

$$\boxed{\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H}\,.$$

Equivariant noncommutative join construction

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group (H, Δ) acting freely on a unital C*-algebra A, we define its equivariant join with H to be the unital C*-algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0,1], A) \underset{\min}{\otimes} H \;\middle|\; f(0) \in \mathbb{C} \otimes H,\ f(1) \in \delta(A) \right\}.$$

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Theorem (P. F. Baum, K. De Commer, P. M. H.)

*The *-homomorphism*

$$\mathrm{id} \otimes \Delta \colon\; C([0,1], A) \underset{\min}{\otimes} H \longrightarrow C([0,1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

defines a free action of the compact quantum group (H, Δ) on the equivariant join C-algebra $A \circledast^{\delta} H$.*

Noncommutative Borsuk-Ulam-type conjectures

Conjecture 1

Let A be a unital C*-algebra with a free action of a non-trivial compact quantum group (H,Δ). Then there does not exist an H-equivariant *-homomorphism $A \to A \circledast^{\delta} H$. (Known to hold for (H,Δ) with classical torsion.)

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Conjecture 2

Let A be a unital C*-algebra with a free action of a non-trivial compact quantum group (H, Δ). If A admits a character, then there does not exist an H-equivariant *-homomorphism $H \to A \circledast^{\delta} H$. (Follows from Conjecture 1.)

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Classical cases

If X is a compact Hausdorff principal G-bundle, $A = C(X)$ and $H = C(G)$, then Conjecture 2 states that the principal G-bundle $X * G$ is not trivializable unless G is trivial. This is clearly true because otherwise $G * G$ would be trivializable, which is tantamount to G being contractible, and the only contractible compact Hausdorff group is the trivial one.

Iterated joins of the quantum $SU(2)$ group

Consider the fibration defining the quaternionic projective space:

$$SU(2) * \cdots * SU(2) \cong S^{4n+3}, \quad S^{4n+3}/SU(2) = \mathbb{H}P^n.$$

To obtain a q-deformation of this fibration, we take $H := C(SU_q(2))$ and $A := C(S_q^{4n+3})$ equal to the n-times iterated equivariant join of H. The quantum principal $SU_q(2)$-bundle thus given is *not* trivializable:

Theorem

*There does not exist a $C(SU_q(2))$-equivariant *-homomorphism*

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This theorem holds because $SU_q(2)$ has classical torsion elements. It also follows from the stable non-triviality of the tautological quaternionic line bundle:

The tautological quaternionic line bundle

If f existed, there would exist an equivariant map F

$$C(SU_q(2)) \to C(S_q^{4n+3}) \circledast^{\delta} C(SU_q(2)) \to C(SU_q(2)) \circledast^{\Delta} C(SU_q(2)).$$

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Furthermore, for any finite-dimensional representation V of a compact quantum group (H, Δ), the associated finitely generated projective module $(H \circledast^{\Delta} H) \square_H V$ is represented by a Milnor idempotent $p_{U^{-1}}$, where U is a matrix of the representation V. Hence an even index pairing calculation for $p_{U^{-1}}$ might be replaced by an odd index pairing calculation for U.

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Now, for $H := C(SU_q(2))$ and V the fundamental representation of $SU_q(2)$, the module $(H \circledast^{\Delta} H) \square_H V$ is the section module of the tautological quaternionic line bundle. It is *not* stably free by the non-vanishing of an index paring of the fundamental representation of $SU_q(2)$ with an appropriate odd Fredholm module. This contradicts the existence of F.