NONCOMMUTATIVE BORSUK-ULAM-TYPE CONJECTURES REVISITED

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Borsuk-Ulam-type conjectures

Conjecture (Baum, Dąbrowski, Hajac)

Let A be a unital C*-algebra with a free action $\delta : A \to A \otimes_{\min} H$ of a non-trivial compact quantum group (H, Δ), and let $A \circledast^{\delta} H$ be the equivariant noncommutative join C*-algebra of A and H with the induced free action of (H, Δ). Then

$$\boxed{\nexists \text{ an } H\text{-equivariant } *\text{-homomorphism } A \longrightarrow A \circledast^{\delta} H}.$$

Furthermore, if A admits a character, then

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Furthermore, if A admits a character, then

$$\boxed{\nexists \text{ an } H\text{-equivariant } *\text{-homomorphism } H \longrightarrow A \circledast^{\delta} H}.$$

Note that the second statement is an immediate consequence of the first one. Indeed, if α is a character on A, then

$$(\alpha \otimes \mathrm{id}_H) \circ \delta \colon A \longrightarrow H$$

is an H-equvariant *-homomorphism.

Compact quantum groups

Definition (S. L. Woronowicz)

A compact quantum group is a unital C^*-algebra H with a given unital $*$-homorphism $\Delta\colon H \longrightarrow H \otimes_{\min} H$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\quad\Delta\quad} & H \underset{\min}{\otimes} H \\ \Big\downarrow{\scriptstyle\Delta} & & \Big\downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ H \underset{\min}{\otimes} H & \xrightarrow[\mathrm{id}\otimes\Delta]{} & H \underset{\min}{\otimes} H \underset{\min}{\otimes} H \end{array}$$

commutes and the two-sided cancellation property holds:

$$\{(a \otimes 1)\Delta(b) \mid a, b \in H\}^{\mathrm{cls}} = H \underset{\min}{\otimes} H = \{\Delta(a)(1 \otimes b) \mid a, b \in H\}^{\mathrm{cls}}.$$

Here "cls" stands for "closed linear span".

Actions of compact quantum groups

Let A be a unital C*-algebra and $\delta : A \to A \otimes_{\min} H$ an injective unital $*$-homomorphism. We call δ an action of the compact quantum group (H, Δ) on A (or a coaction of H on A) iff

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H$ (counitality).

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Given a compact quantum group (H, Δ), we denote by $\mathcal{O}(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations.

The Peter-Weyl subalgebra

of A is $\mathcal{P}_H(A) := \{\, a \in A \,|\, \delta(a) \in A \otimes_{\mathrm{alg}} \mathcal{O}(H) \,\}$.

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Definition (D. A. Ellwood)

A coaction δ is called free iff

$$\boxed{\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H}\,.$$

Equivariant noncommutative joins

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group (H, Δ) acting freely on a unital C*-algebra A, we define its equivariant join with H to be the unital C*-algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0,1], A) \underset{\min}{\otimes} H \;\middle|\; f(0) \in \mathbb{C} \otimes H,\ f(1) \in \delta(A) \right\}.$$

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Theorem (P. F. Baum, K. De Commer, P. M. H.)

*The *-homomorphism*

$$\mathrm{id} \otimes \Delta \colon\; C([0,1], A) \underset{\min}{\otimes} H \longrightarrow C([0,1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

defines a free action of the compact quantum group (H, Δ) on the equivariant join C-algebra $A \circledast^{\delta} H$.*

Torsion noncommutative Borsuk-Ulam theorem

Theorem (main)

Let A be a unital C-algebra with a free action $\delta : A \to A \otimes_{\min} H$ of a non-trivial compact quantum group (H, Δ), and let $A \circledast^{\delta} H$ be the equivariant noncommutative join C*-algebra of A and H with the induced free action of (H, Δ). Then, if H admits a character different from the counit whose finite convolution power is the counit, the following statements are true and equivalent:*

1. *$\nexists$ an H-equivariant $*$-homomorphism $A \longrightarrow A \circledast^{\delta} H$.*
2. *$\nexists$ a $*$-homomorphism $\gamma : A \longrightarrow \mathcal{C}A$ such that $\mathrm{ev}_1 \circ \gamma = id_A$.*

Here $\mathcal{C}A := A \circledast \mathbb{C}$ is the cone of A, and ev_1 is the evaluation at 1.

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The first statement can be reduced to its special case $H = C(\mathbb{Z}/k\mathbb{Z})$, $k > 1$, proven by B. Passer. The second statement is follows from the first one for any compact quantum group (H, Δ), and the reverse implicationis true when (H, Δ) admits the counit. For $A = C(S^n)$ the second statement is *equivalent* to the Brouwer fixed-point theorem for the ball B^n.

K-lemma

Lemma

Let $\mathcal{C}B$ and ΣB be respectively the cone and the unreduced suspension of a unital C-algebra B admitting a character. Then*

$$K_0(\Sigma B) \cong \mathbb{Z}[1] \oplus K_1(B).$$

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Proof. Since ΣB is a surjective pullback of $\mathcal{C}B$ with $\mathcal{C}B$ over B, we get the 6-term exact sequence:

$$\begin{array}{ccccc} K_0(\Sigma B) & \longrightarrow & K_0(\mathcal{C}B) \oplus K_0(\mathcal{C}B) & \longrightarrow & K_0(B) \\ \uparrow & & & & \downarrow \\ K_1(B) & \longleftarrow & K_1(\mathcal{C}B) \oplus K_1(\mathcal{C}B) & \longleftarrow & K_1(\Sigma B) \end{array} .$$

Hence we obtain the exact sequence

$$0 \longrightarrow K_1(B) \longrightarrow K_0(\Sigma B) \longrightarrow \mathbb{Z} \longrightarrow 0$$

proving the lemma. □

Associated-vector-bundle theorem

Theorem

Let G be a compact connected semisimple Lie group. Then, there exists a finite-dimensional representation V of G such that for any compact Hausdorff space X equipped with a free G-action, the associated vector bundle

$$(X * G) \overset{G}{\times} V$$

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To prove the theorem, we first show it for $X = G$ using the K-lemma with $B = C(G)$, then take a G-equivariant map $G * G \to X * G$, and apply the following observation:

Pulling back classical bundles

Let G be a compact Hausdorff group acting on compact Hausdorff spaces Y and Y', and let $F : Y' \to Y$ be an equivariant continuous map. Then, if the G-action on Y is free, so is the G-action on Y', and the formula

$$Y' \ni p \longmapsto ([p], F(p)) \in Y'/G \underset{Y/G}{\times} Y$$

defines a G-equivariant homeomorphism of compact principal bundles.

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Corollary

If V is a representation of G, the following associated vector bundles over Y'/G are isomorphic

$$(F|_{Y'/G})^* \left(Y \overset{G}{\times} V \right) \cong Y' \overset{G}{\times} V \,.$$

In particular, if $\dim V < \infty$, the induced map $(F|_{Y'/G})^* : K^0(Y) \to K^0(Y')$ satisfies

$$(F|_{Y'/G})^* \left(\left[Y \overset{G}{\times} V \right] \right) = \left[Y' \overset{G}{\times} V \right] .$$

Deformation theorem

Theorem

Let G be a compact connected semisimple Lie group. Let $(C(G_q), \Delta_q)$, $q > 0$, be a family of compact quantum groups that is a q-deformation of $(C(G), \Delta)$. Then, for any $q > 0$ there exists a finite-dimensional left $\mathcal{O}(G_q)$-comodule V_q such that for any unital C-algebra A admitting a character and equipped with a free action of $(C(G_q), \Delta_q)$, the associated finitely generated projective left $(A \circledast_\delta C(G_q))^{\mathrm{co}\, C(G_q)}$-module*

$$\mathcal{P}_{C(G_q)}\big(A \underset{\delta}{\circledast} C(G_q)\big)\Box V_q$$

is not stably free.

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$$\mathcal{P}_{C(G_q)}\big(A \underset{\delta}{\circledast} C(G_q)\big)\Box V_q$$

*is **not** stably free.*

Note that this theorem implies the Borsuk-Ulam conjecture of type II for q-deformations of compact connected semisimple Lie groups. As in the classical case, we first prove it for $A = C(G_q)$, use a character on A to construct an H-equivariant *-homomorphism $A \circledast C(G_q) \longrightarrow C(G_q) \circledast C(G_q)$, and apply:

Noncommutative pulling-back theorem

Theorem (P. M. H. and T. Maszczyk)

Let (H,Δ) be a compact quantum group, C and C' (H,Δ)-C-algebras, B and B' the corresponding fixed-point subalgebras, and $f: C \to C'$ an equivariant *-homomorphism. Then, if the action of (H,Δ) on C is free and V is a representation of (H,Δ), the following left B'-modules are isomorphic*

$$B'_f \underset{B}{\otimes} \big(\mathcal{P}_H(C)\Box V\big) \cong \mathcal{P}_H(C')\Box V\,.$$

Here B'_f stands for the B'-B-bimodule with the right action given by f, i.e. $b\cdot c = bf(c)$.

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Corollary

The induced map $(f|_B)_* : K_0(B) \to K_0(B')$ satisfies

$$\boxed{(f|_B)_*\big([\mathcal{P}_H(C)\Box V]\big) = [\mathcal{P}_H(C')\Box V]}\,.$$