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서울대학교

SEOUL NATIONAL UNIVERSITY

**FROM THE NON-CONTRACTIBILITY OF  
COMPACT QUANTUM GROUPS  
TO A NONCOMMUTATIVE  
BORSUK-ULAM-TYPE CONJECTURE**

**Piotr M. Hajac (IMPAN)**

Joint work with P. F. Baum, L. Dąbrowski and S. Neshveyev.

# Passanger announcement

This talk has been modified from its original version. It has been formatted to fit the conference schedule. To see the original version, please come on 5 July 2017 at 14:15 to a mathematical sciences colloquium at the University of Oulu, Finland.

# Classical contractibility

A topological space  $X$  is **contractible** iff there exists a retraction of its inclusion into its cone:

$$\exists \varphi: CX \rightarrow X \quad \text{s.t.} \quad X \hookrightarrow CX \xrightarrow{\varphi} X \quad \text{is} \quad \text{id}_X.$$

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If we replace  $X$  by a topological group  $G$ , then the contractibility of  $G$  can be phrased equivariantly:

$$\exists \varphi: CG \rightarrow G \quad \text{s.t.} \quad G \hookrightarrow CG \xrightarrow{\varphi} G \quad \text{is} \quad G\text{-equivariant.}$$

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The only contractible compact group is the trivial one. Hence, if  $G$  is a non-trivial compact group, then  $\gamma$  does **not** exist. How about  $G^{*n+1} \longrightarrow G^{*n}$  ?

# Join formulation and classical generalization

## Theorem (Borsuk-Ulam)

Let  $n$  be a positive natural number. There does **not** exist a  $\mathbb{Z}/2\mathbb{Z}$ -equivariant continuous map  $S^{n-1} * \mathbb{Z}/2\mathbb{Z} \rightarrow S^{n-1}$ .

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This naturally leads to:

## A classical Borsuk-Ulam-type conjecture

Let  $X$  be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group  $G$ . Then, for the diagonal action of  $G$  on  $X * G$ , there does **not** exist a  $G$ -equivariant continuous map  $f : X * G \rightarrow X$ .

Note that the existence of  $f$  is equivalent to the existence of  $\varphi : CX \rightarrow X$  s.t.  $X \hookrightarrow CX \xrightarrow{\varphi} X$  is  $G$ -equivariant.



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Note that the existence of  $f$  is equivalent to the existence of  $\varphi : CX \rightarrow X$  s.t.  $X \hookrightarrow CX \xrightarrow{\varphi} X$  is  $G$ -equivariant. At the moment, **the conjecture is known to hold under the assumption of local triviality**. In its full generality, it is deeply related to the celebrated Hilbert-Smith conjecture.

# The contractibility of compact quantum spaces

Let  $A$  be a unital  $C^*$ -algebra. We call  $A$  **unittally contractible** iff there exists a  $*$ -homomorphism

$$\phi : A \rightarrow \mathcal{C}A := \{x \in C([0, 1], A) \mid \text{ev}_0(x) \in \mathbb{C}\} \text{ s.t. } \text{ev}_1 \circ \phi = \text{id}_A.$$

If  $A$  is unittally contractible, then it admits a character and its K-theory coincides with the K-theory of  $\mathbb{C}$ .

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If  $(H, \Delta)$  is a compact quantum group, then its contractibility can be phrased equivariantly:

$$\exists \phi : H \rightarrow \mathcal{C}H \quad \text{s.t.} \quad H \xrightarrow{\phi} \mathcal{C}H \xrightarrow{\text{ev}_1} H \quad \text{is } H\text{-equivariant.}$$

Indeed, the equivalence follows from the fact that all equivariant morphisms from a locally compact quantum group to itself are  $*$ -isomorphisms (R. Meyer, S. Roy, S. L. Woronowicz).

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It follows from the classification of finite dimensional  $C^*$ -algebras that **the only contractible finite quantum group is the trivial one.**

# Equivariant noncommutative join construction

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group  $(H, \Delta)$  acting freely on a unital  $C^*$ -algebra  $A$ , we define its **equivariant join** with  $H$  to be the unital  $C^*$ -algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0, 1], A) \underset{\min}{\otimes} H \mid f(0) \in \mathbb{C} \otimes H, f(1) \in \delta(A) \right\}.$$

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Theorem (P. F. Baum, K. De Commer, P. M. H.)

*The  $\ast$ -homomorphism*

$$\text{id} \otimes \Delta: C([0, 1], A) \underset{\min}{\otimes} H \longrightarrow C([0, 1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

*defines a free action of the compact quantum group  $(H, \Delta)$  on the equivariant join  $C^*$ -algebra  $A \overset{\delta}{\ast} H$ .*

# Noncommutative Borsuk-Ulam-type conjecture

## Conjecture

Let  $A$  be a unital  $C^*$ -algebra with a free action of a non-trivial compact quantum group  $(H, \Delta)$ . Then there **does not exist an  $H$ -equivariant  $*$ -homomorphism  $A \rightarrow A \otimes^{\delta} H$** .

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## Theorem (torsion case)

Let  $A$  be a unital  $C^*$ -algebra with a free action  $\delta : A \rightarrow A \otimes_{\min} H$  of a non-trivial compact quantum group  $(H, \Delta)$ , and let  $A \otimes_{\delta} H$  be the equivariant noncommutative join  $C^*$ -algebra of  $A$  and  $H$  with the induced free action of  $(H, \Delta)$ . Then, **if  $H$  admits a character different from the counit whose finite convolution power is the counit**, the following statements are true and equivalent:

- 1  $\nexists$  an  $H$ -equivariant  $*$ -homomorphism  $A \rightarrow A \otimes_{\delta} H$ .
- 2  $\nexists$  a  $*$ -homomorphism  $\phi : A \rightarrow \mathcal{C}A$  such that  $\text{ev}_1 \circ \gamma$  is  $H$ -equivariant.



## Final remarks

The first statement can be reduced to its special case

$H = C(\mathbb{Z}/k\mathbb{Z})$ ,  $k > 1$ , proven by B. Passer. The second statement follows from the first one for any compact quantum group  $(H, \Delta)$ , and the reverse implication is true when  $(H, \Delta)$  admits the counit.

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Since the Toeplitz  $C^*$ -algebra  $\mathcal{T}$  admits a free  $\mathbb{Z}/2\mathbb{Z}$ -action, it follows from the above theorem that  $\mathcal{T}$  is not unitaly contractible despite having the same K-theory as  $\mathbb{C}$ , and despite being a deformation of a contractible space (disc)  $C^*$ -algebra.

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From the special case  $(A, \delta) = (H, \Delta)$  we can conclude that any compact quantum group with a torsion character is not contractible. Furthermore, (a stronger version of) the first statement holds if the compact quantum group  $(H, \Delta)$  admits a finite-dimensional representation whose  $K_1$ -class is not trivial. Hence it holds for the abelian compact quantum groups  $C_r^*F_n$ , which do not admit the counit. (Here  $F_n$  is the free group on  $n$  generators.)