PULLING BACK NONCOMMUTATIVE ASSOCIATED VECTOR BUNDLES AND CONSTRUCTING QUANTUM QUATERNIONIC PROJECTIVE SPACES

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Pulling back classical bundles

Let G be a compact Hausdorff group acting on compact Hausdorff spaces Y and Y', and let $F : Y' \to Y$ be an equivariant continuous map. Then, if the G-action on Y is free, so is the G-action on Y', and the formula

$$Y' \ni p \longmapsto ([p], F(p)) \in Y'/G \underset{Y/G}{\times} Y$$

defines a G-equivariant homeomorphism of compact principal bundles.

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Corollary

If V is a representation of G, the following associated vector bundles over Y'/G are isomorphic

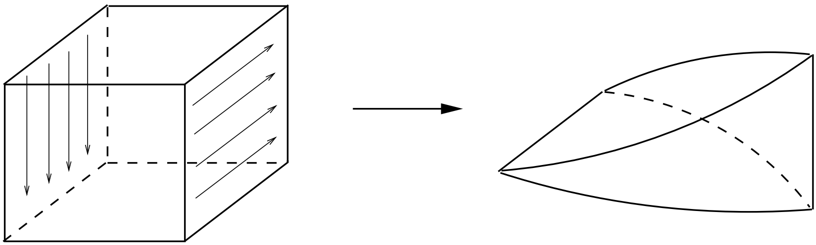
$$(F|_{Y'/G})^* \big(Y \underset{G}{\times} V\big) \cong Y' \underset{G}{\times} V \,.$$

In particular, if $\dim V < \infty$, the induced map $(F|_{Y'/G})^* : K^0(Y) \to K^0(Y')$ satisfies

$$(F|_{Y'/G})^* \big([Y \underset{G}{\times} V]\big) = [Y' \underset{G}{\times} V] \,.$$

Equivariant join construction

For any topological spaces X and Y, one defines the join space $X * Y$ as the quotient of $[0, 1] \times X \times Y$ by a certain equivalence relation:



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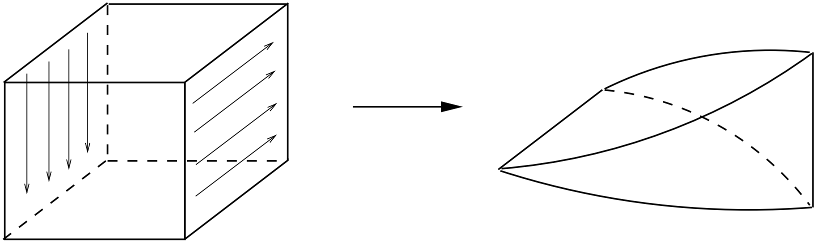

If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G, then the diagonal action of G on the join $X * G$ is again continuous and free.

Gauged equivariant join construction

If $Y = G$, we can construct the join G-space $X * Y$ in a different manner: at 0 we collapse $X \times G$ to G as before, and at 1 we collapse $X \times G$ to $(X \times G)/R_D$ instead of X. Here R_D is the equivalence relation generated by

$$\boxed{(x, h) \sim (x', h'), \text{ where } xh = x'h'}\,.$$

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More precisely, let R'_J be the equivalence relation on $I \times X \times G$ generated by

$$(0, x, h) \sim (0, x', h) \quad \text{and} \quad (1, x, h) \sim (1, x', h'),\ \text{where } xh = x'h'.$$

The formula $[(t, x, h)]k := [(t, x, hk)]$ defines a continuous right G-action on $(I \times X \times G)/R'_J$, and the formula

$$X * G \ni [(t, x, h)] \longmapsto [(t, xh^{-1}, h)] \in (I \times X \times G)/R'_J$$

yields a G-equivariant homeomorphism.

Classical projective spaces

Consider the n-th iteration:

$$(\mathbb{Z}/2\mathbb{Z}) * \cdots * (\mathbb{Z}/2\mathbb{Z}) \cong S^n.$$

With the diagonal $\mathbb{Z}/2\mathbb{Z}$-action, we obtain

$$S^n/(\mathbb{Z}/2\mathbb{Z}) = \mathbb{RP}^n.$$

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Consider the n-th iteration:

$$SU(2) * \cdots * SU(2) \cong S^{4n+3}.$$

With the diagonal $SU(2)$-action, we obtain

$$S^{4n+3}/SU(2) = \mathbb{HP}^n.$$

Compact quantum group

Definition (S. L. Woronowicz)

A compact quantum group is a unital C^*-algebra H with a given unital $*$-homorphism $\Delta\colon H \longrightarrow H \otimes_{\min} H$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\quad\Delta\quad} & H \underset{\min}{\otimes} H \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \Delta\otimes\mathrm{id}} \\ H \underset{\min}{\otimes} H & \xrightarrow[\mathrm{id}\otimes\Delta]{} & H \underset{\min}{\otimes} H \underset{\min}{\otimes} H \end{array}$$

commutes and the two-sided cancellation property holds:

$$\{(a\otimes 1)\Delta(b) \mid a, b \in H\}^{\mathrm{cls}} = H \underset{\min}{\otimes} H = \{\Delta(a)(1\otimes b) \mid a, b \in H\}^{\mathrm{cls}}.$$

Here "cls" stands for "closed linear span".

Free actions of compact quantum groups

Let A be a unital C^*-algebra and $\delta : A \to A \otimes_{\min} H$ an injective unital $*$-homomorphism. We call δ a coaction of H on A (or an action of the compact quantum group (H, Δ) on A) if

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H$ (counitality).

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Given a compact quantum group (H, Δ), we denote by $\mathcal{O}(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations.

The Peter-Weyl subalgebra

of A is $\mathcal{P}_H(A) := \{\, a \in A \,|\, \delta(a) \in A \otimes_{\mathrm{alg}} \mathcal{O}(H) \,\}$.

The Peter-Weyl-Galois Theorem

Theorem (P. F. Baum, K. De Commer, P.M.H.)

Let A be a unital C-algebra equipped with an action of a compact quantum group (H, Δ). The following conditions are equivalent:*

1. *The action is free.*
2. *The action satisfies the Peter-Weyl-Galois condition.*
3. *The action is strongly monoidal.*

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Put $B = A^{\mathrm{co}H} := \{a \in A \mid \delta(a) = a \otimes 1\}$ (coaction-invariants).

The Peter-Weyl-Galois condition

is the bijectivity of the canonical map
$\mathcal{P}_H(A) \otimes_B \mathcal{P}_H(A) \ni x \otimes y \xmapsto{can} (x \otimes 1)\delta(y) \in \mathcal{P}_H(A) \otimes_{\mathrm{alg}} \mathcal{O}(H)$.

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Let V and W be $\mathcal{O}(H)$-comodules (representations of (H, Δ)).

The strong monoidality

is the bijectivity of the natural map
$(\mathcal{P}_H(A) \Box V) \otimes_B (\mathcal{P}_H(A) \Box W) \longrightarrow \mathcal{P}_H(A) \Box (V \otimes_{\mathrm{alg}} W)$.

Main result

Theorem

Let (H,Δ) be a compact quantum group, A and A' (H,Δ)-C-algebras, B and B' the corresponding fixed-point subalgebras, and $f : A \to A'$ an equivariant *-homomorphism. Then, if the action of (H,Δ) on A is free and V is a representation of (H,Δ), the following left B'-modules are isomorphic*

$$B'_f \underset{B}{\otimes} \big(\mathcal{P}_H(A)\Box V\big) \cong \mathcal{P}_H(A')\Box V\,.$$

Here B'_f stands for the B'-B-bimodule with the right action given by f, i.e. $b \cdot c = bf(c)$.

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Corollary

The induced map $(f|_B)_* : K_0(B) \to K_0(B')$ satisfies

$$(f|_B)_*\big([\mathcal{P}_H(A)\Box V]\big) = [\mathcal{P}_H(A')\Box V]\,.$$

Equivariant noncommutative join

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group (H, Δ) acting freely on a unital C*-algebra A, we define its equivariant join with H to be the unital C*-algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0,1], A) \underset{\min}{\otimes} H \;\middle|\; f(0) \in \mathbb{C} \otimes H,\ f(1) \in \delta(A) \right\}.$$

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*The *-homomorphism*

$$\mathrm{id} \otimes \Delta\colon\ C([0,1], A) \underset{\min}{\otimes} H \longrightarrow C([0,1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

defines a free action of the compact quantum group (H, Δ) on the equivariant join C-algebra $A \circledast^{\delta} H$.*

Iterated joins of the quantum $SU(2)$ group

To obtain a q-deformation of

$$S^{4n+3}/SU(2) = \mathbb{HP}^n,$$

we take $H := C(SU_q(2))$ and $A := C(S_q^{4n+3})$ equal to the n-times iterated equivariant join of H. We view the fixed-point subalgebra $C(S_q^{4n+3})^{SU_q(2)}$ as the defining C*-algebra $C(\mathbb{HP}_q^n)$ of a *quantum quaternionic projective space*.

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Then we define the noncommutative tautological quaternionic line bundle and its dual as noncommutative complex vector bundles associated through the contragredient representation $V_f^\vee$ of the fundamental represention of $SU_q(2)$ and the fundamental represention V_f itself, respectively.

Quantum quaternionic line bundles

Theorem

For any $n \in \mathbb{N} \setminus \{0\}$ and $0 < q \leq 1$, the noncommutative tautological quaternionic line bundle and its dual are not stably trivial as noncommutative complex vector bundles, i.e., the finitely generated projective left $C(\mathbb{HP}_q^n)$-modules $\mathcal{P}_{SU_q(2)}(S_q^{4n+3})\Box V_f^\vee$ and $\mathcal{P}_{SU_q(2)}(S_q^{4n+3})\Box V_f$ are not stably free.

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<u>**Proof outline:**</u> There exists an $SU_q(2)$-equivariant *-homomorphism $C(S_q^{4n+3}) \to C(SU_q(2)) \circledast^{\Delta} C(SU_q(2)) =: C(S_q^7)$. Hence, by the main theorem, it suffices to prove that the left $C(\mathbb{HP}_q^1)$-modules $\mathcal{P}_{SU_q(2)}(S_q^7)\square V_f^\vee$ and $\mathcal{P}_{SU_q(2)}(S_q^7)\square V_f$ are not stably free.

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Quantum quaternionic principal bundles

Let (H, Δ) be a compact quantum group acting freely on a unital C*-algebra A. It follows from Hopf-Galois theory that, if there exists an H-equivariant *-homomorphism $H \to A$, then the associated A^{coH}-module $\mathcal{P}_H(A)\Box V$ is *free* for any left $\mathcal{O}(H)$-comodule V.

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Consequently, the quantum principal $SU_q(2)$-bundle $S_q^{4n+3} \to \mathbb{HP}_q^n$ is *not* trivializable:

Corollary

*There does not exist a $C(SU_q(2))$-equivariant *-homomorphism*

$$f \colon C(SU_q(2)) \longrightarrow C(S_q^{4n+3}).$$