FIELDS

GENERATING NONCOMMUTATIVE VECTOR BUNDLES OVER QUANTUM COMPLEX PROJECTIVE PLANES

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Joint works with F. D'Andrea, C. Farsi, A. Kaygun, T. Maszczyk, M. Tobolski, B. Zieliński.

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Path algebra

Definition

A quiver $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of the set of vertices Q_0, the set of arrows Q_1, and the source and target maps $s, t\colon Q_1 \to Q_0$ assigning to each arrow its source and target vertex respectively.

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Definition

Let k be a field and Q a quiver. The path algebra kQ is the k-algebra whose underlying vector space has as its basis the set of all finite paths (compositions of arrows and vertices) and whose product is given by the composition of paths if the end of the first path matches the beginning of the second path, and that is zero otherwise.

Leavitt path algebra

Let k be a field and Q a quiver. Elements of the set $Q_1^* := \{x^* \mid x \in Q_1\}$ are called ghost arrows, and the extended quiver $\widehat{Q} := (Q_0, Q_1 \coprod Q_1^*, \hat{s}, \hat{t})$ is defined by

$$\hat{s}(x) := s(x), \quad \hat{s}(x^*) := t(x), \quad \hat{t}(x) := t(x), \quad \hat{t}(x^*) := s(x).$$

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Definition

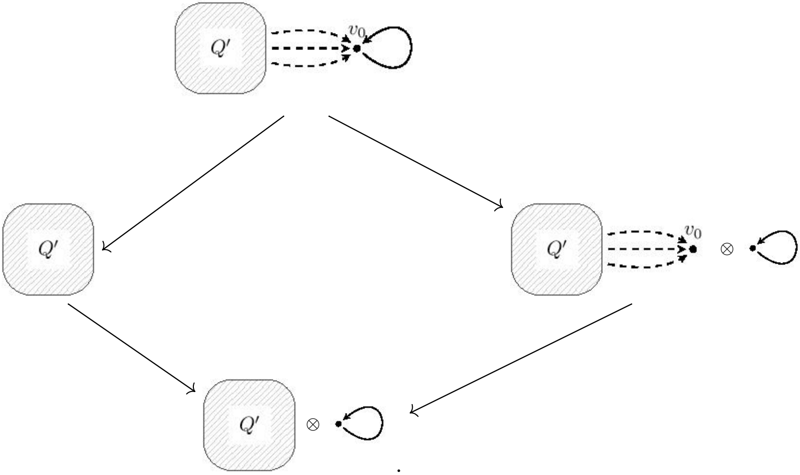
The Leavitt path algebra $L_k(Q)$ of a quiver Q is the path algebra of the extended quiver $\widehat{Q}$ divided by the ideal generated by the relations:

1. $\forall\, x_i\,, x_j \in Q_1 :\; x_i^* x_j = \delta_{ij} t(x_i)$,
2. $\forall\, v \in Q_0$ such that the preimage $s^{-1}(v)$ is not empty and finite:

$$\sum_{x \in s^{-1}(v)} x x^* = v.$$

From Leavitt algebras to pullback algebras

Let Q be a finite quiver consisting of a sub-quiver Q' emitting arrows to the external vertex v_0 whose only outgoing arrow is a loop arrow. Then, if all Q'-emitted arrows begin in a vertex emitting an arrow ending inside the sub-quiver Q', for an appropriate choice of algebra homorphisms, the following diagram D is commutative:



Graph C*-algebra

Definition

Let Q be a quiver. The universal C*-algebra $C^*(Q)$ of the quiver Q is generated by elements of $Q_0 \cup Q_1$ subject to the relations:

1. $\forall\, v \in Q_0 :\ v^* = v$,
2. $\forall\, v_i\,, v_j \in Q_0 :\ v_i v_j = \delta_{ij} v_i$,
3. $\forall\, x_i\,, x_j \in Q_1 :\ x_i^* x_j = \delta_{ij} t(x_i)$,
4. $\forall\, v \in Q_0$ such that the preimage $s^{-1}(v)$ is not empty and finite: $\sum_{x \in s^{-1}(v)} x x^* = v$,
5. $\forall\, x \in Q_1 :\ x x^* \leqslant s(x)$.

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Using the fact that for any element c of a C*-algebra we have

$$c^* c = 0 \quad \Rightarrow \quad 0 = \sqrt{\|c^* c\|} = \|c\| \quad \Rightarrow \quad c = 0,$$

one can prove that the graph C*-algebra of a quiver Q enjoys all path-algebraic relations of the extended quiver path algebra $\mathbb{C}\widehat{Q}$. The key step is to show that

$$\boxed{\forall\, x \in Q_1 :\ s(x)x = x = xt(x)}\,.$$

Quantum balls and spheres

Choosing the ground field to be $\mathbb{C}$ and completing Leavitt path algebras to graph C*-algebras, the commutative diagram D becomes a commutative diagram of $U(1)$-equivariant *-homomorphisms. Moreover, taking the quiver Q to be the graph representing a Vaksman-Soibelman quantum sphere, we obtain:

Theorem

$\forall\, n \in \mathbb{N} \setminus \{0\}\ \exists$ *a* $U(1)$*-equivariant pullback of C*-algebras:*

$$\begin{array}{ccc} & C(S_q^{2n+1}) & \\ \swarrow & & \searrow \\ C(S_q^{2n-1}) & & C(B_q^{2n}) \otimes C(S^1) \\ \searrow & & \swarrow \\ & C(S_q^{2n-1}) \otimes C(S^1). & \end{array}$$

Bundles over quantum complex projective spaces

Corollary

$\forall\, n \in \mathbb{N} \setminus \{0\}\ \exists$ *a pullback of C*-algebras:*

$$\begin{array}{ccc} & C(\mathbb{C}P_q^{n}) & \\ \swarrow & & \searrow \\ C(\mathbb{C}P_q^{n-1}) & & C(B_q^{2n}) \\ \searrow & & \swarrow \\ & C(S_q^{2n-1}). & \end{array}$$

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\end{array}
$$

$$
\begin{array}{ccccc}
K_0(C(\mathbb{C}P_q^n)) & \longrightarrow & K_0(C(\mathbb{C}P_q^{n-1})) \oplus K_0(C(B_q^{2n})) & \longrightarrow & K_0(C(S_q^{2n-1})) \\
\uparrow{\scriptstyle \partial_{10}} & & & & \downarrow{\scriptstyle \partial_{01}} \\
K_1(C(S_q^{2n-1})) & \longleftarrow & K_1(C(\mathbb{C}P_q^{n-1})) \oplus K_1(C(B_q^{2n})) & \longleftarrow & K_1(C(\mathbb{C}P_q^n))
\end{array}
$$

Multipullback quantum spheres S_H^{2N+1}

$C(S_H^{2N+1})$ is the C*-subalgebra of $\prod_{i=0}^{N} \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i}$ defined by the compatibility conditions prescribed by the following diagrams ($0 \leq i < j \leq N$, $\otimes$-supressed):

$$\begin{array}{ccc} \mathcal{T}^i C(S^1) \mathcal{T}^{N-i} & & \mathcal{T}^j C(S^1) \mathcal{T}^{N-j} \\ & \searrow^{\sigma_j} \qquad {}^{\sigma_i}\swarrow & \\ & \mathcal{T}^i C(S^1) \mathcal{T}^{j-i-1} C(S^1) \mathcal{T}^{N-j}. & \end{array}$$

Here $\sigma_k := \mathrm{id}^k \otimes \sigma \otimes \mathrm{id}^{N-k}$ with domains and codomains determined by the context.

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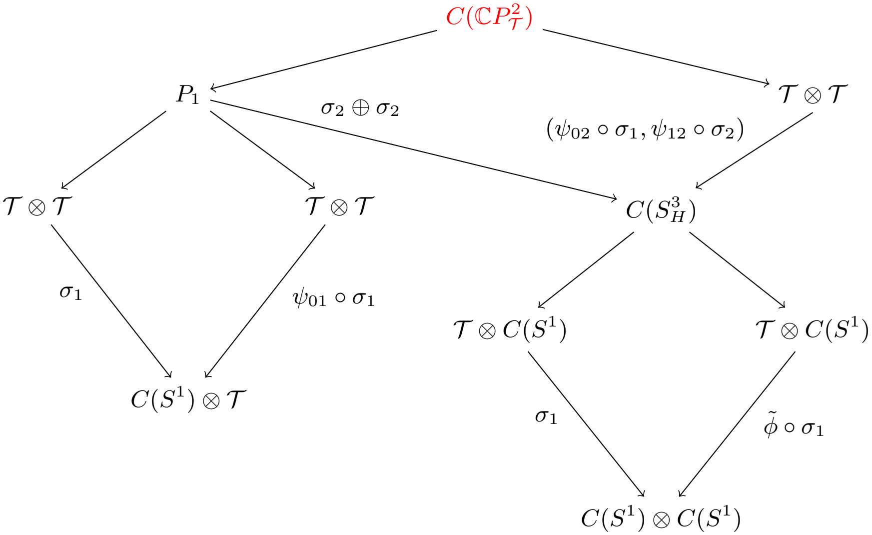
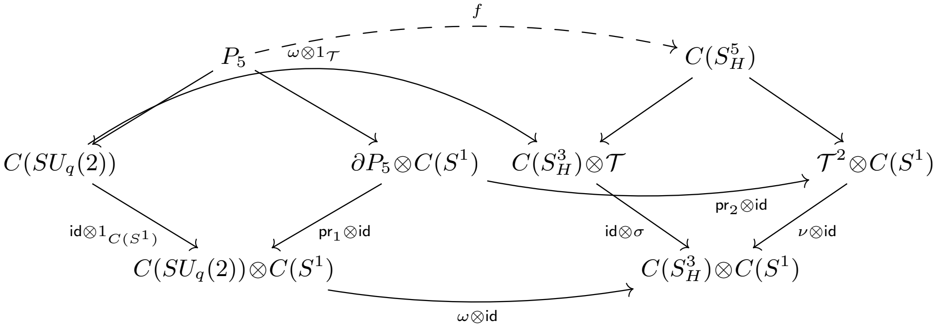
We equip all C*-algebras in the diagrams with the diagonal actions of $U(1)$. Since all morphisms in the diagrams are $U(1)$-equivariant, we obtain the diagonal $U(1)$-action on $C(S_H^{2N+1})$.

Multipullback quantum complex projective



Reducing to the quantum-group case



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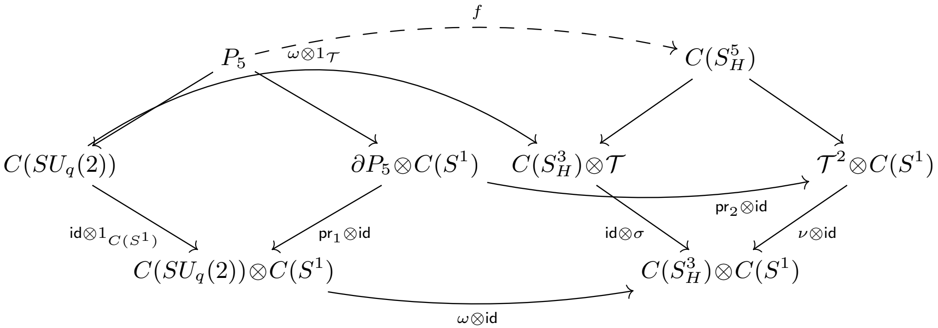

K-Isomorphism Lemma

The above *-homomorphisms are $U(1)$-equivariant, and the induced *-homomorphisms on fixed-point subalgebras yield isomorphisms on K-groups.

Main result

Definition

Let $k \in \mathbb{Z}$. We call the left $C(\mathbb{C}P^2_{\mathcal{T}})$-module

$$L_k := \{a \in C(S^5_H) \mid \forall\, \lambda \in U(1) : \alpha_\lambda(a) = \lambda^k a\}$$

the section module of the associated line bundle of winding number k.

Theorem

The group $K_0(C(\mathbb{C}P^2_{\mathcal{T}}))$ is freely generated by elements

$$[1], \quad [L_1] - [1], \quad [L_1 \oplus L_{-1}] - [2].$$

Furthermore, $L_1 \oplus L_{-1} \cong C(\mathbb{C}P^2_{\mathcal{T}}) \oplus C(\mathbb{C}P^2_{\mathcal{T}})e$. Here $e \in C(\mathbb{C}P^2_{\mathcal{T}})$ is an idempotent such that $C(\mathbb{C}P^2_{\mathcal{T}})e$ cannot be realized as a finitely generated projective module associated with the $U(1)$-C-algebra $C(S^5_H)$ of Heegaard quantum 5-sphere.*

New Geometry of Quantum Dynamics

The Banach Center, Warsaw, 15 January – 19 January 2018.

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