NONCOMMUTATIVE BORSUK–ULAM TYPE CONJECTURES

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Trieste, February 2017

The Borsuk-Ulam Theorem

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Let n be a positive natural number. If $f : S^n \to \mathbb{R}^n$ is continuous, then there exists a pair $(p, -p)$ of antipodal points on S^n such that $f(p) = f(-p)$.

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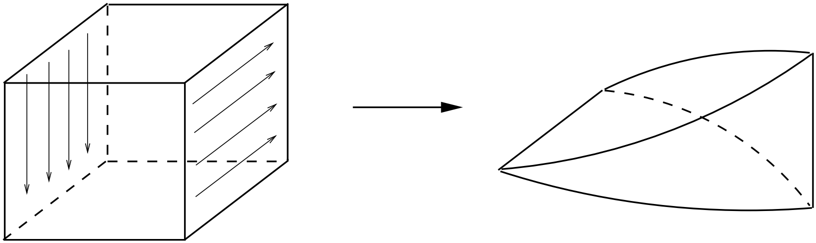
Theorem (join formulation)

Let n be a positive natural number. There does not exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^{n-1} * \mathbb{Z}/2\mathbb{Z} \to S^{n-1}$.

Equivariant join construction

For any topological spaces X and Y, one defines the join space $X * Y$ as the quotient of $[0, 1] \times X \times Y$ by a certain equivalence relation:



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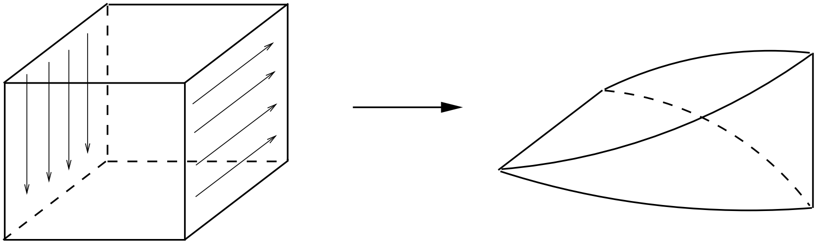

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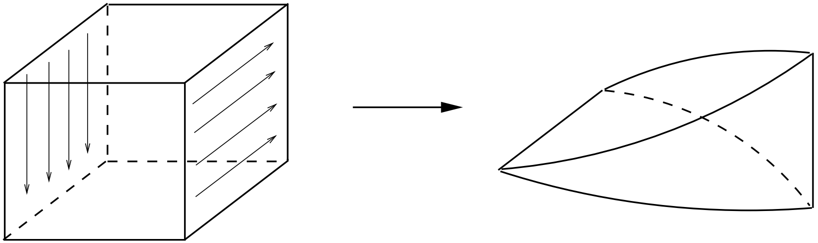

If X is a compact Hausdorff space with a continuous free action of a (locally) compact Hausdorff group G, then the diagonal action of G on the join $X * G$ is again continuous and free. In particular, for the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on S^{n-1}, we obtain a $\mathbb{Z}/2\mathbb{Z}$-equivariant identification $S^n \cong S^{n-1} * \mathbb{Z}/2\mathbb{Z}$ for the antipodal and diagonal actions respectively.

Index of G-spaces for finite groups

Let us use the notation $E_n G := \underbrace{G * ... * G}_{n+1}$.

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Definition (G-space index)

Let G be a finite group, $|G| > 1$ and let X be a G-space. We define

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- $ind_G(X * Y) \leq ind_G(X) + ind_G(Y) + 1$*,*
- $ind_G(X * G) = ind_G(X) + 1$*.*

Join formulation and classical generalization

The join formulation of Borsuk-Ulam theorem naturally leads to:

Conjecture (P. Baum, L. Dąbrowski, P. M. Hajac)

Let X be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group G. Then, for the diagonal action of G on $X * G$, there does not exist a G-equivariant continuous map $f : X * G \to X$.

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Corollary to a corollary

The weaker version of the Hilbert-Smith conjecture.

Free actions on C*-algebras

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Definition

Let $\delta : A \to A \otimes C(G)$ be an injective unital *-homomorphism. We call such δ a coaction of $C(G)$ on A if we have that

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity)
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\ h \in C(G)\}^{\mathrm{cls}} = A \otimes C(G)$ (counitality)

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Definition (D. A. Ellwood)

We say that the coaction δ is free when

$$\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\mathrm{cls}} = A \otimes C(G).$$

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Suppose a compact abelian group G acts on a unital C*-algebra A via $\alpha : G \to Aut(A)$. This gives rise to homogeneous subspaces $A_\tau = \{a \in A : \alpha_g(a) = \tau(g)\ \forall g \in G\}$, which are defined for any $\tau \in \hat{G}$, the Pontryagin dual of G. The action is free if and only if for each $\tau \in \hat{G}$, $\overline{A_\tau A A_\tau^*} = A$.

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Example 2

Let $G = \mathbb{Z}/p\mathbb{Z}$ for a prime number p. Then $\widehat{\mathbb{Z}/p\mathbb{Z}} = \mathbb{Z}/p\mathbb{Z}$, given by the pth roots of unity. Then $A_n = \{a \in A : \alpha(a) = e^{2\pi i n/p} a\}$.

Noncommutative equivariant joins

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Definition

Let δ be a coaction of $C(G)$ on A. Then we define the noncommutative join of A and $C(G)$ as:

$$A \circledast C(G) = \left\{ f \in C([0,1], A \otimes C(G)) \middle| f(0) \in C(G), f(1) \in A \right\}.$$

We can make the above join into a noncommutative G-space with the diagonal action.

Rephrasing classical results

Notation: $\mathbb{Z}/p := \mathbb{Z}/p\mathbb{Z}$.

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Lemma

Let $\mathbb{Z}/k$, $k \geq 2$, act freely on itself by multiplication. Equip $\mathbb{C}$ with the $\mathbb{Z}/k$ action, $z \mapsto e^{2\pi i/k}z$. If $n \geq 2$ and $f_1, ..., f_n \in C(E_{2n}\mathbb{Z}/k)$ are equivariant for these actions, then $f_1, ..., f_n$ have a common zero.

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The above lemma is proved by using:

Theorem (Dold)

Let G be a finite group with $|G| > 1$. Let X be an n-connected G-space, and let Y be a free paracompact G-space of dimension at most n. Then there is no equivariant map $X \to Y$.

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Some more properties of the index used in the proof:

- if X is n-connected, then $ind_G(X)$, then $ind_G(X) \geq n+1$,
- if Y is a free paracompact G-space of dimension n, then $ind_G(Y) \leq n$.

Main theorem

Theorem (B. Passer)

Let A a unital C-algebra. If $\delta : A \to A \otimes C(G)$ is a free coaction for a compact Hausdorff group G with torsion, then there does not exist an equivariant unital C*-algebra homomorphism $A \to A \circledast C(G)$.*

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Proof outline: If G has torsion elements, then the problem can be reduced to $\mathbb{Z}/p$ action on A.
We assume that such equivariant map exists and apply it in succession producing a chain

$$A \to A \circledast C(G) \to A \circledast C(G) \circledast C(G) \to \ldots$$

of equivariant maps.

Main theorem

For any n, we obtain an equivariant unital *-homomorphism

$$\phi_n : A \to \underbrace{C(\mathbb{Z}/k) \circledast \ldots \circledast C(\mathbb{Z}/k)}_{n+1} \simeq C(E_n\mathbb{Z}/k).$$

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The action is saturated, therefore for a given $\tau \in \widehat{\mathbb{Z}/k}$ there is a finite m and $a_1, \dots, a_m, b_1, \dots, b_m \in A_\tau$, such that $\sum a_i b_i^*$ is invertible in A.

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Then for $m = n$, we have that $\phi_{2n}(a_1), ..., \phi_{2n}(a_n) \in C(E_n\mathbb{Z}/k)$ do not have common zeros, which contradicts the previous lemma.

Thanks!