



UNIwersytet im. Adama Mickiewicza  
w Poznaniu

**THERE AND BACK AGAIN:  
FROM THE BORSUK-ULAM THEOREM  
TO QUANTUM SPACES**

**Piotr M. Hajac** (IMPAN / University of New Brunswick)

**Tatiana Shulman** (IMPAN)

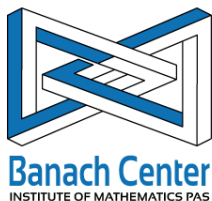
Based on joint work of Piotr M. Hajac with  
Paul F. Baum, Ludwik Dąbrowski and Tomasz Maszczyk

Jiří Matoušek

# Using the Borsuk-Ulam Theorem

Lectures on Topological Methods  
in Combinatorics and Geometry



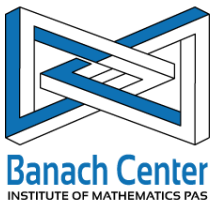


SIMONS  
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1 Sep – 30 Nov 2016, Simons Semester in the Banach Center

**NONCOMMUTATIVE GEOMETRY THE NEXT GENERATION**

*Paul F. Baum, Alan Carey, Piotr M. Hajac, Tomasz Maszczyk*



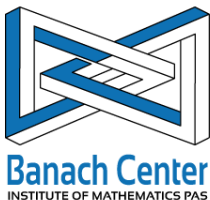
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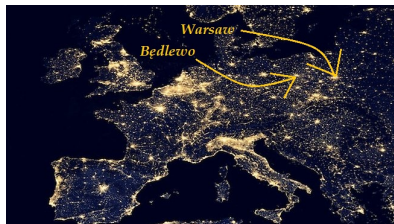
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4–17 September, Będlewo & Warsaw, Master Class on:

## Noncommutative geometry and quantum groups

- 1 **Cyclic homology**  
by Masoud Khalkhali and Ryszard Nest
- 2 **Noncommutative index theory**  
by Nigel Higson and Erik Van Erp
- 3 **Topological quantum groups and Hopf algebras**  
by Alfons Van Daele and Stanisław L. Woronowicz
- 4 **Structure and classification of  $C^*$ -algebras**  
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19 September – 14 October, 20-hour lecture courses:

- 1 **An invitation to  $C^*$ -algebras** by Karen R. Strung
- 2 **An invitation to Hopf algebras** by Réamonn Ó Buachalla
- 3 **Noncommutative topology for beginners** by Tatiana Shulman

- ① 17–21 Oct. **Cyclic homology**  
J. Cuntz, P. M. Hajac, T. Maszczyk, R. Nest



# Conferences

- ① 17–21 Oct. **Cyclic homology**  
J. Cuntz, P. M. Hajac, T. Maszczyk, R. Nest
- ② 24–28 Oct. **Noncommutative index theory**  
P. F. Baum, A. Carey, M. J. Pflaum, A. Sitarz

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- 4 21–25 Nov. **Structure and classification of  $C^*$ -algebras**  
G. Elliott, K. R. Strung, W. Winter, J. Zacharias

**GEOMETRY, REPRESENTATION THEORY  
AND THE BAUM-CONNES CONJECTURE**

A workshop in honour of **Paul F. Baum** on the occasion of his 80th birthday organized by Alan Carey, George Elliott, Piotr M. Hajac, and Ryszard Nest.

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Sponsored by:

- The Fields Institute, University of Toronto, Canada
- National Science Foundation, USA
- The Pennsylvania State University, USA



FIELDS



# What is a compact quantum group?

Definition (S. L. Woronowicz)

A **compact quantum group** is a unital  $C^*$ -algebra  $H$  with a given unital  $*$ -homomorphism  $\Delta: H \rightarrow H \otimes_{\min} H$  such that the diagram

$$\begin{array}{ccc}
 H & \xrightarrow{\Delta} & H \otimes_{\min} H \\
 \downarrow \Delta & & \downarrow \Delta \otimes \text{id} \\
 H \otimes_{\min} H & \xrightarrow{\text{id} \otimes \Delta} & H \otimes_{\min} H \otimes_{\min} H
 \end{array}$$

commutes and the two-sided cancellation property holds:

$$\{(a \otimes 1)\Delta(b) \mid a, b \in H\}^{\text{cls}} = H \otimes_{\min} H = \{\Delta(a)(1 \otimes b) \mid a, b \in H\}^{\text{cls}}.$$

Here “cls” stands for “closed linear span”.

# Free actions of compact quantum groups

Let  $A$  be a unital  $C^*$ -algebra and  $\delta : A \rightarrow A \otimes_{\min} H$  a unital  $*$ -homomorphism. We call  $\delta$  a **coaction** of  $H$  on  $A$  (or an action of the compact quantum group  $(H, \Delta)$  on  $A$ ) iff

- 1  $(\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \Delta) \circ \delta$  (coassociativity),
- 2  $\{\delta(a)(1 \otimes h) \mid a \in A, h \in H\}^{\text{cls}} = A \otimes_{\min} H$  (counitality)
- 3  $\ker \delta = 0$  (injectivity).

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Definition (D. A. Ellwood)

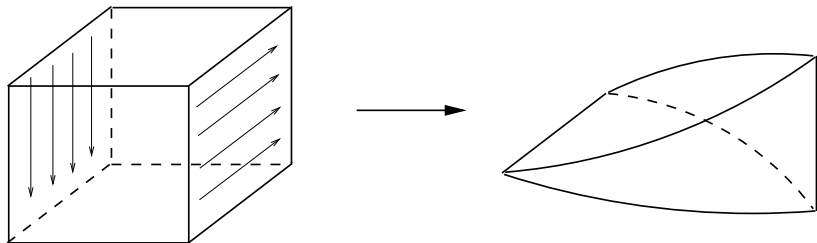
A coaction  $\delta$  is called **free** iff

$$\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\text{cls}} = A \otimes_{\min} H .$$



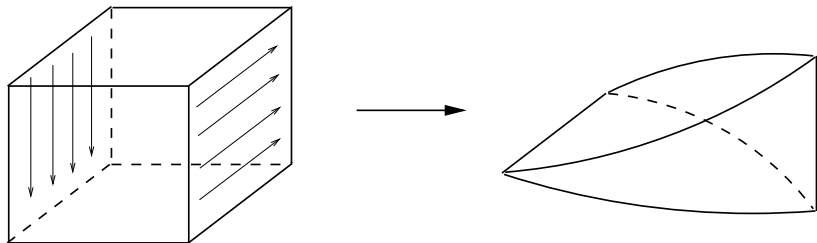
# Equivariant join construction

For any topological spaces  $X$  and  $Y$ , one defines the **join** space  $X * Y$  as the quotient of  $[0, 1] \times X \times Y$  by a certain equivalence relation:



# Equivariant join construction

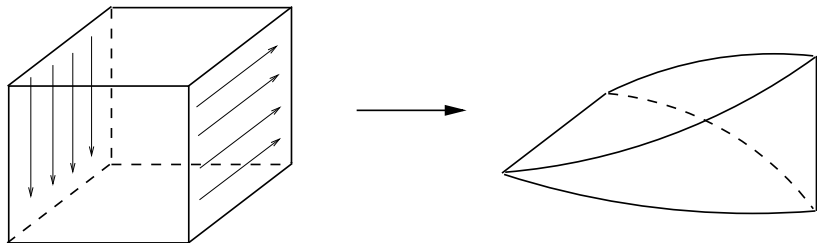
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If  $X$  is a compact Hausdorff space with a continuous free action of a compact Hausdorff group  $G$ , then the diagonal action of  $G$  on the join  $X * G$  is again continuous and free.

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If  $X$  is a compact Hausdorff space with a continuous free action of a compact Hausdorff group  $G$ , then the diagonal action of  $G$  on the join  $X * G$  is again continuous and free. In particular, for the antipodal action of  $\mathbb{Z}/2\mathbb{Z}$  on  $S^{n-1}$ , we obtain a  $\mathbb{Z}/2\mathbb{Z}$ -equivariant identification  $S^n \cong S^{n-1} * \mathbb{Z}/2\mathbb{Z}$  for the antipodal and diagonal actions respectively.

## Gauged equivariant join construction

If  $Y = G$ , we can construct the join  $G$ -space  $X * Y$  in a different manner: at 0 we collapse  $X \times G$  to  $G$  as before, and at 1 we collapse  $X \times G$  to  $(X \times G)/R_D$  instead of  $X$ . Here  $R_D$  is the equivalence relation generated by

$$\boxed{(x, h) \sim (x', h'), \text{ where } xh = x'h'}.$$

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More precisely, let  $R'_J$  be the equivalence relation on  $I \times X \times G$  generated by

$$(0, x, h) \sim (0, x', h) \quad \text{and} \quad (1, x, h) \sim (1, x', h'), \text{ where } xh = x'h'.$$

The formula  $[(t, x, h)]k := [(t, x, hk)]$  defines a continuous right  $G$ -action on  $(I \times X \times G)/R'_J$ , and the formula

$$X * G \ni [(t, x, h)] \longmapsto [(t, xh^{-1}, h)] \in (I \times X \times G)/R'_J$$

yields a  $G$ -equivariant homeomorphism.

# Join formulation and classical generalization

Thus the Borsuk-Ulam Theorem is equivalent to:

Theorem (join formulation)

*Let  $n$  be a positive natural number. There does **not** exist a  $\mathbb{Z}/2\mathbb{Z}$ -equivariant continuous map  $\tilde{f}: S^{n-1} * \mathbb{Z}/2\mathbb{Z} \rightarrow S^{n-1}$ .*

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This naturally leads to:

## A classical Borsuk-Ulam-type conjecture

Let  $X$  be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group  $G$ . Then, for the diagonal action of  $G$  on  $X * G$ , there does **not** exist a  $G$ -equivariant continuous map  $f: X * G \rightarrow X$ .

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## Corollary

*Ageev's conjecture about the Menger compacta.*

# Equivariant noncommutative join construction

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group  $(H, \Delta)$  acting freely on a unital  $C^*$ -algebra  $A$ , we define its **equivariant join** with  $H$  to be the unital  $C^*$ -algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0, 1], A) \underset{\min}{\otimes} H \mid f(0) \in \mathbb{C} \otimes H, f(1) \in \delta(A) \right\}.$$

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Theorem (P. F. Baum, K. De Commer, P. M. H.)

*The  $\ast$ -homomorphism*

$$\text{id} \otimes \Delta: C([0, 1], A) \underset{\min}{\otimes} H \longrightarrow C([0, 1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

*defines a free action of the compact quantum group  $(H, \Delta)$  on the equivariant join  $C^*$ -algebra  $A \overset{\delta}{\ast} H$ .*

# Noncommutative Borsuk-Ulam-type conjectures

## Conjecture 1

Let  $A$  be a unital nuclear  $C^*$ -algebra with a free action of a non-trivial compact quantum group  $(H, \Delta)$ . Then there **does not exist** an  $H$ -equivariant  $*$ -homomorphism  $A \rightarrow A \otimes^{\delta} H$ .

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## Conjecture 2

Let  $A$  be a unital nuclear  $C^*$ -algebra with a free action of a non-trivial compact quantum group  $(H, \Delta)$ . Then there **does not exist an  $H$ -equivariant  $*$ -homomorphism  $H \rightarrow A \otimes^{\delta} H$ .**

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## The classical cases

If  $X$  is a compact Hausdorff principal  $G$ -bundle,  $A = C(X)$  and  $H = C(G)$ , then Conjecture 2 states that the principal  $G$ -bundle  $X * G$  is not trivializable unless  $G$  is trivial. This is clearly true because otherwise  $G * G$  would be trivializable, which is tantamount to  $G$  being contractible, and the only contractible compact Hausdorff group is trivial.

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# Iterated joins of the quantum $SU(2)$ group

Consider the fibration defining the quaternionic projective space:

$$SU(2) * \cdots * SU(2) \cong S^{4n+3}, \quad S^{4n+3}/SU(2) = \mathbb{H}P^n.$$

To obtain a  $q$ -deformation of this fibration, we take  $H := C(SU_q(2))$  and  $A := C(S_q^{4n+3})$  equal to the  $n$ -times iterated equivariant join of  $H$ . The quantum principal  $SU_q(2)$ -bundle thus given is *not* trivializable:

## Theorem (main)

There does **not** exist a  $C(SU_q(2))$ -equivariant  $*$ -homomorphism  $f: C(SU_q(2)) \rightarrow C(S_q^{4n+3}) \otimes^\delta C(SU_q(2))$ .



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**Proof outline:** If  $f$  existed, there would be an equivariant map  $F: C(SU_q(2)) \rightarrow C(S_q^{4n+3}) \otimes^\delta C(SU_q(2)) \rightarrow C(SU_q(2)) \otimes^\Delta C(SU_q(2))$ . Furthermore, for any finite-dimensional representation  $V$  of a compact quantum group  $(H, \Delta)$ , the associated finitely-generated projective module  $(H \otimes^\Delta H) \square_H V$  is represented by a Milnor idempotent  $p_{U^{-1}}$ , where  $U$  is a matrix of the representation  $V$ . If  $H := C(SU_q(2))$  and  $V$  is the fundamental representation of  $SU_q(2)$ , then  $(H \otimes^\Delta H) \square_H V$  is not stably free by an index pairing calculation. This contradicts the existence of  $F$ .  $\square$

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