

11 May 2016



THERE AND BACK AGAIN: FROM THE BORSUK-ULAM THEOREM TO QUANTUM SPACES

Piotr M. Hajac (IMPAN / University of New Brunswick)
Tatiana Shulman (IMPAN)

Based on joint work of Piotr M. Hajac with
Paul F. Baum, Ludwik Dąbrowski and Tomasz Maszczyk

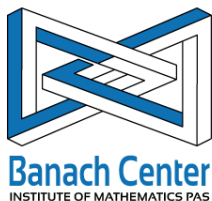
Jiří Matoušek

Using the Borsuk-Ulam Theorem

Lectures on Topological Methods
in Combinatorics and Geometry



Banach-Simons Semester



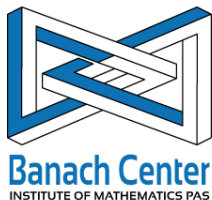
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1 Sep – 30 Nov 2016, Simons Semester in the Banach Center

NONCOMMUTATIVE GEOMETRY THE NEXT GENERATION

Paul F. Baum, Alan Carey, Piotr M. Hajac, Tomasz Maszczyk

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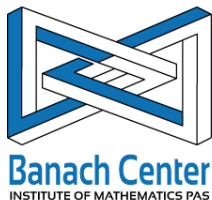
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- ⑧ 21–25 Nov. **Structure and classification of C^* -algebras**
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18–22 July 2016, the Fields Institute

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- The Fields Institute, University of Toronto, Canada
- National Science Foundation, USA
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FIELDS



What is a compact quantum group?

Definition (S. L. Woronowicz)

A **compact quantum group** is a unital C^* -algebra H with a given unital $*$ -homomorphism $\Delta: H \rightarrow H \otimes_{\min} H$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes_{\min} H \\ \downarrow \Delta & & \downarrow \Delta \otimes \text{id} \\ H \otimes_{\min} H & \xrightarrow{\text{id} \otimes \Delta} & H \otimes_{\min} H \otimes_{\min} H \end{array}$$

commutes and the two-sided cancellation property holds:

$$\{(a \otimes 1)\Delta(b) \mid a, b \in H\}^{\text{cls}} = H \otimes_{\min} H = \{\Delta(a)(1 \otimes b) \mid a, b \in H\}^{\text{cls}}.$$

Here “cls” stands for “closed linear span”.

Free actions of compact quantum groups

Let A be a unital C^* -algebra and $\delta : A \rightarrow A \otimes_{\min} H$ a unital $*$ -homomorphism. We call δ a **coaction** of H on A (or an action of the compact quantum group (H, Δ) on A) iff

- ① $(\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \Delta) \circ \delta$ (coassociativity),
- ② $\{\delta(a)(1 \otimes h) \mid a \in A, h \in H\}^{\text{cls}} = A \otimes_{\min} H$ (counitality)
- ③ $\ker \delta = 0$ (injectivity).

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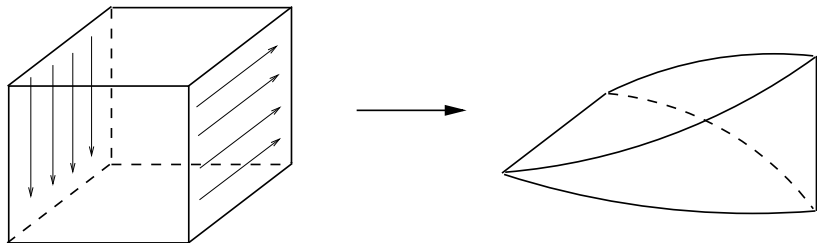
Definition (D. A. Ellwood)

A coaction δ is called **free** iff

$$\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\text{cls}} = A \otimes_{\min} H .$$

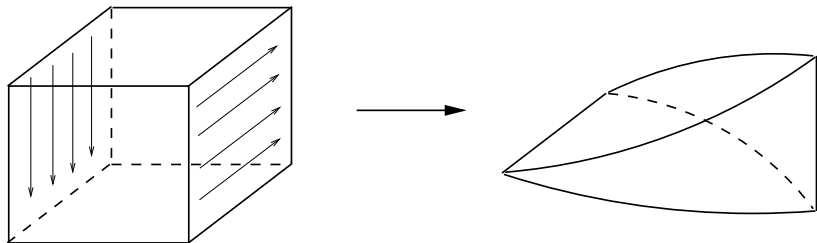
Equivariant join construction

For any topological spaces X and Y , one defines the **join** space $X * Y$ as the quotient of $[0, 1] \times X \times Y$ by a certain equivalence relation:



Equivariant join construction

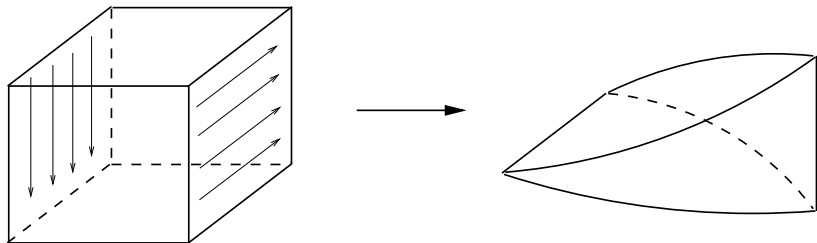
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If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G , then the diagonal action of G on the join $X * G$ is again continuous and free.

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If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G , then the diagonal action of G on the join $X * G$ is again continuous and free. In particular, for the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on S^{n-1} , we obtain a $\mathbb{Z}/2\mathbb{Z}$ -equivariant identification $S^n \cong S^{n-1} * \mathbb{Z}/2\mathbb{Z}$ for the antipodal and diagonal actions respectively.

Gauged equivariant join construction

If $Y = G$, we can construct the join G -space $X * Y$ in a different manner: at 0 we collapse $X \times G$ to G as before, and at 1 we collapse $X \times G$ to $(X \times G)/R_D$ instead of X . Here R_D is the equivalence relation generated by

$$\boxed{(x, h) \sim (x', h'), \text{ where } xh = x'h'}.$$

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More precisely, let R'_J be the equivalence relation on $I \times X \times G$ generated by

$$(0, x, h) \sim (0, x', h) \quad \text{and} \quad (1, x, h) \sim (1, x', h'), \text{ where } xh = x'h'.$$

The formula $[(t, x, h)]k := [(t, x, hk)]$ defines a continuous right G -action on $(I \times X \times G)/R'_J$, and the formula

$$X * G \ni [(t, x, h)] \longmapsto [(t, xh^{-1}, h)] \in (I \times X \times G)/R'_J$$

yields a G -equivariant homeomorphism.

Join formulation and classical generalization

Thus the Borsuk-Ulam Theorem is equivalent to:

Theorem (join formulation)

*Let n be a positive natural number. There does **not** exist a $\mathbb{Z}/2\mathbb{Z}$ -equivariant continuous map $\tilde{f}: S^{n-1} * \mathbb{Z}/2\mathbb{Z} \rightarrow S^{n-1}$.*

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This naturally leads to:

A classical Borsuk-Ulam-type conjecture

Let X be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group G . Then, for the diagonal action of G on $X * G$, there does **not** exist a G -equivariant continuous map $f: X * G \rightarrow X$.

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Corollary

Ageev's conjecture about the Menger compacta.

Equivariant noncommutative join construction

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group (H, Δ) acting freely on a unital C^* -algebra A , we define its **equivariant join** with H to be the unital C^* -algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0, 1], A) \underset{\min}{\otimes} H \mid f(0) \in \mathbb{C} \otimes H, f(1) \in \delta(A) \right\}.$$

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Theorem (P. F. Baum, K. De Commer, P. M. H.)

The \ast -homomorphism

$$\text{id} \otimes \Delta: C([0, 1], A) \underset{\min}{\otimes} H \longrightarrow C([0, 1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

defines a free action of the compact quantum group (H, Δ) on the equivariant join C^ -algebra $A \overset{\delta}{\ast} H$.*

Noncommutative Borsuk-Ulam-type conjectures

Conjecture 1

Let A be a unital nuclear C^* -algebra with a free action of a non-trivial compact quantum group (H, Δ) . Then there **does not exist an H -equivariant $*$ -homomorphism $A \rightarrow A \otimes^{\delta} H$.**

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The classical cases

If X is a compact Hausdorff principal G -bundle, $A = C(X)$ and $H = C(G)$, then Conjecture 2 states that the principal G -bundle $X * G$ is not trivializable unless G is trivial. This is clearly true because otherwise $G * G$ would be trivializable, which is tantamount to G being contractible, and the only contractible compact Hausdorff group is trivial.

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Iterated joins of the quantum $SU(2)$ group

Consider the fibration defining the quaternionic projective space:

$$SU(2) * \cdots * SU(2) \cong S^{4n+3}, \quad S^{4n+3}/SU(2) = \mathbb{H}P^n.$$

To obtain a q -deformation of this fibration, we take $H := C(SU_q(2))$ and $A := C(S_q^{4n+3})$ equal to the n -times iterated equivariant join of H . The quantum principal $SU_q(2)$ -bundle thus given is *not* trivializable:

Theorem (main)

There does **not** exist a $C(SU_q(2))$ -equivariant $*$ -homomorphism $f: C(SU_q(2)) \rightarrow C(S_q^{4n+3}) \otimes^\delta C(SU_q(2))$.

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Proof outline: If f existed, there would be an equivariant map $F: C(SU_q(2)) \rightarrow C(S_q^{4n+3}) \otimes^\delta C(SU_q(2)) \rightarrow C(SU_q(2)) \otimes^\Delta C(SU_q(2))$. Furthermore, for any finite-dimensional representation V of a compact quantum group (H, Δ) , the associated finitely-generated projective module $(H \otimes^\Delta H) \square_H V$ is represented by a Milnor idempotent $p_{U^{-1}}$, where U is a matrix of the representation V . If $H := C(SU_q(2))$ and V is the fundamental representation of $SU_q(2)$, then $(H \otimes^\Delta H) \square_H V$ is not stably free by an index pairing calculation. This contradicts the existence of F . \square

References

- *Noncommutative Borsuk-Ulam-type conjectures*;
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Banach Center Publications 106 (2015), 9–18.

References

- *Noncommutative Borsuk-Ulam-type conjectures*;
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