PULLING BACK ASSOCIATED NONCOMMUTATIVE VECTOR BUNDLES AND CONSTRUCTING QUANTUM QUATERNIONIC PROJECTIVE SPACES

Piotr M. Hajac (IMPAN / University of New Brunswick)

Joint work with Tomasz Maszczyk

16 March 2016, the EU capital

Free actions of compact quantum groups

Let A be a unital C^*-algebra and $\delta : A \to A \otimes_{\min} H$ an injective unital $*$-homomorphism. We call δ a coaction of H on A (or an action of the compact quantum group (H, Δ) on A) if

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H$ (counitality).

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A coaction δ is called free iff

$$\boxed{\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H}.$$

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Given a compact quantum group (H, Δ), we denote by $\mathcal{O}(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations.

The Peter-Weyl subalgebra

of A is $\mathcal{P}_H(A) := \{\, a \in A \,|\, \delta(a) \in A \otimes_{\mathrm{alg}} \mathcal{O}(H) \,\}$.

The Peter-Weyl-Galois Theorem

Theorem (P. F. Baum, K. De Commer, P.M.H.)

Let A be a unital C-algebra equipped with an action of a compact quantum group (H, Δ). The following conditions are equivalent:*

1. *The action is free.*
2. *The action satisfies the Peter-Weyl-Galois condition.*
3. *The action is strongly monoidal.*

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Put $B = A^{\mathrm{co}H} := \{a \in A \mid \delta(a) = a \otimes 1\}$ (coaction-invariants).

The Peter-Weyl-Galois condition

is the bijectivity of the canonical map
$\mathcal{P}_H(A) \otimes_B \mathcal{P}_H(A) \ni x \otimes y \stackrel{can}{\longmapsto} (x \otimes 1)\delta(y) \in \mathcal{P}_H(A) \otimes_{\mathrm{alg}} \mathcal{O}(H)$.

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Let V and W be $\mathcal{O}(H)$-comodules (representations of (H,Δ)).

The strong monoidality

is the bijectivity of the natural map
$(\mathcal{P}_H(A) \Box V) \otimes_B (\mathcal{P}_H(A) \Box W) \longrightarrow \mathcal{P}_H(A) \Box (V \otimes_{\mathrm{alg}} W)$.

18–22 July 2016, the Fields Institute

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- The Fields Institute, University of Toronto, Canada
- National Science Foundation, USA
- The Pennsylvania State University, USA

Pulling Back Theorem

Theorem

Let (H, Δ) be a compact quantum group, A and A' (H, Δ)-C-algebras, B and B' the corresponding fixed-point subalgebras, and $f : A \to A'$ an equivariant *-homomorphism. Then, if the action of (H, Δ) on A is free and V is a representation of (H, Δ), the following left B'-modules are isomorphic*

$$B'_f \underset{B}{\otimes} \big(\mathcal{P}_H(A)\Box V\big) \cong \mathcal{P}_H(A')\Box V \,.$$

Here B'_f stands for the B'-B-bimodule with the right action given by f, i.e. $b \cdot c = bf(c)$.

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Corollary

The induced map $(f|_B)_* : K_0(B) \to K_0(B')$ satisfies

$$\boxed{(f|_B)_*\left([\mathcal{P}_H(A)\Box V]\right) = [\mathcal{P}_H(A')\Box V]}\,.$$

Strong connections

Let $\mathcal{H}$ be a Hopf algebra with bijective antipode S and $\mathcal{P}$ a right $\mathcal{H}$-comodule algebra for a coaction $\delta : \mathcal{P} \to \mathcal{P} \otimes_{\mathrm{alg}} \mathcal{H}$. We view $\mathcal{H}$ as an $\mathcal{H}$-bicomodule via its comultiplication. We consider $\mathcal{P} \otimes_{\mathrm{alg}} \mathcal{P}$ as an $\mathcal{H}$-bicomodule via

$$\begin{aligned} \mathsf{id} \otimes \delta &\qquad \text{right coaction,} \\ \Big((S^{-1} \otimes \mathsf{id}) \circ \mathrm{flip} \circ \delta\Big) \otimes \mathsf{id} &\qquad \text{left coaction.} \end{aligned}$$

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A strong connection is a unital bicolinear map $\ell : \mathcal{H} \to \mathcal{P} \otimes_{\mathrm{alg}} \mathcal{P}$ such that *multiplication* $\circ\, \ell = \varepsilon$.

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Theorem (T. Brzeziński, P.M.H.)

Let $\mathcal{B}$ be the coaction-invariant subalgebra. The existence of a strong connection is equivalent to the bijectivity of the canonical map $\mathcal{P} \otimes_{\mathcal{B}} \mathcal{P} \to \mathcal{P} \otimes_{\mathrm{alg}} \mathcal{H}$ and the existence of a left $\mathcal{B}$-linear right $\mathcal{H}$-colinear splitting of the multiplication map $\mathcal{B} \otimes \mathcal{P} \to \mathcal{P}$ (equivariant projectivity).

The Chern-Galois proof

Note first that, since $\mathcal{O}(H)$ is cosemisimple, any comodule is a direct sum of finite-dimensional comodules, so that it suffices to prove the theorem for finite-dimesional representations of (H, Δ).

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Note first that, since $\mathcal{O}(H)$ is cosemisimple, any comodule is a direct sum of finite-dimensional comodules, so that it suffices to prove the theorem for finite-dimesional representations of (H, Δ).

Furthermore, by the PWG Theorem and the cosemisimplicity of $\mathcal{O}(H)$, there exists a strong connection

$$\ell : \mathcal{O}(H) \longrightarrow \mathcal{P}_H(A) \otimes \mathcal{P}_H(A)$$

on $\mathcal{P}_H(A)$. Next, the equivariance of the *-homomorphism f implies that $\ell' := (f \otimes f) \circ \ell$ is a strong connection on $\mathcal{P}_H(A')$.

The Chern-Galois proof

Now take advantage of Chern-Galois theory to show that applying f componentwise to an idempotent matrix over B representing $\mathcal{P}_H(A)\Box V$ through ℓ is an idempotent matrix over B' of the following block form:

$$e := \begin{pmatrix} e' & 0 \\ r & 0 \end{pmatrix}.$$

Here e' is an idempotent matrix representing $\mathcal{P}_H(A')\Box V$ through ℓ'. It follows from $e^2 = e$ that and $re' = r$.

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Here e' is an idempotent matrix representing $\mathcal{P}_H(A')\Box V$ through ℓ'. It follows from $e^2 = e$ that and $re' = r$. Finally, the computation

$$\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}\begin{pmatrix} e' & 0 \\ r & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} = \begin{pmatrix} e' & 0 \\ 0 & 0 \end{pmatrix}$$

shows that modules represented respectively by e and e' are isomorphic, i.e. $B'_f \underset{B}{\otimes} \big(\mathcal{P}_H(A)\Box V\big) \cong \mathcal{P}_H(A')\Box V$. $\Box$

The faithful-flatness proof

It follows from the first part of the preceding proof that the canonical map

$$\mathcal{P}_H(A') \underset{B'}{\otimes} \mathcal{P}_H(A') \ni x \otimes y \overset{can'}{\longmapsto} (x \otimes 1)\delta'(y) \in \mathcal{P}_H(A') \otimes \mathcal{O}(H)$$

is bijective, and that $\mathcal{P}_H(A')$ is faithfully flat over B'. Consequently,

$$\widetilde{f} := m_{\mathcal{P}_H(A')} \circ (\mathsf{id} \otimes f) \colon B'_f \underset{B}{\otimes} \mathcal{P}_H(A) \longrightarrow \mathcal{P}_H(A')$$

is an isomorphism if and only if

$$\mathsf{id} \otimes \big(m_{\mathcal{P}_H(A')} \circ (\mathsf{id} \otimes f)\big) \colon \mathcal{P}_H(A') \underset{B'}{\otimes} B'_f \underset{B}{\otimes} \mathcal{P}_H(A) \longrightarrow \mathcal{P}_H(A') \underset{B'}{\otimes} \mathcal{P}_H(A')$$

is an isomorphism. It is the case if and only if

$$m_{\mathcal{P}_H(A')} \otimes f \colon \mathcal{P}_H(A')_f \underset{\mathcal{P}_H(A)}{\otimes} \mathcal{P}_H(A) \underset{B}{\otimes} \mathcal{P}_H(A) \longrightarrow \mathcal{P}_H(A') \underset{B'}{\otimes} \mathcal{P}_H(A')$$

is an isomorphism.

The faithful-flatness proof

Thus, from the commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{P}_H(A')_f \underset{\mathcal{P}_H(A)}{\otimes} \mathcal{P}_H(A) \underset{B}{\otimes} \mathcal{P}_H(A) & \xrightarrow{m_{\mathcal{P}_H(A')}\otimes f} & \mathcal{P}_H(A') \underset{B'}{\otimes} \mathcal{P}_H(A') \\
\Big\downarrow{\scriptstyle (m_{\mathcal{P}_H(A')}\otimes \mathsf{id})\circ(\mathsf{id}\otimes can)} & & \Big\downarrow{\scriptstyle can'} \\
\mathcal{P}_H(A') \underset{\text{alg}}{\otimes} \mathcal{O}(H) & \xrightarrow{\qquad \mathsf{id} \qquad} & \mathcal{P}_H(A') \underset{\text{alg}}{\otimes} \mathcal{O}(H)
\end{array}$$

and the bijectivity of the canonical maps, we infer that $\widetilde{f}$ is an isomorphism. Since it is equivariant, we conclude that

$$\widetilde{f} \otimes \mathsf{id} : \left(B'_f \underset{B}{\otimes} \mathcal{P}_H(A)\right)\Box V \longrightarrow \mathcal{P}_H(A')\Box V$$

is an isomorphism of left B'-modules. Finally, as $\mathcal{O}(H)$ is cosemisimple, and any comodule over a cosemisimple Hopf algebra is coflat, it follows that

$$B'_f \underset{B}{\otimes} \left(\mathcal{P}_H(A)\Box V\right) \cong \mathcal{P}_H(A')\Box V\,,$$

as desired. $\Box$

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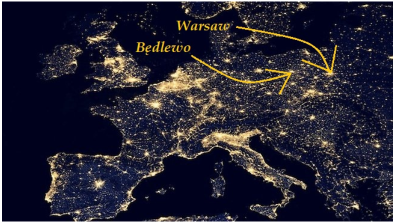

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8. 21–25 Nov. Structure and classification of C*-algebras
   G. Elliott, K. R. Strung, W. Winter, J. Zacharias

Equivariant noncommutative join construction

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group (H, Δ) acting freely on a unital C*-algebra A, we define its equivariant join with H to be the unital C*-algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0,1], A) \underset{\min}{\otimes} H \;\middle|\; f(0) \in \mathbb{C} \otimes H,\ f(1) \in \delta(A) \right\}.$$

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Theorem (P. F. Baum, K. De Commer, P. M. H.)

*The *-homomorphism*

$$\mathrm{id} \otimes \Delta \colon\; C([0,1], A) \underset{\min}{\otimes} H \longrightarrow C([0,1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

defines a free action of the compact quantum group (H, Δ) on the equivariant join C-algebra $A \circledast^{\delta} H$.*

Key Lemma

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Let (H,Δ) be a compact quantum group, A and A' (H,Δ)-C-algebras, and $F : A \to A'$ an equivariant *-homomorphism. Assume that there exists a finite-dimensional representation V of (H,Δ) such that the finitely generated projective module $(A' \circledast^{\delta'} H)\Box V$ is not free. Then there does not exist an equivariant *-homomorphism $H \to A \circledast^{\delta} H$ (Borsuk-Ulam).*

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Proof: Since *-homomorphism F is equivariant, the *-homomorphism

$$\mathrm{id} \otimes F \otimes \mathrm{id} : C([0,1]) \underset{\min}{\otimes} A \underset{\min}{\otimes} H \longrightarrow C([0,1]) \underset{\min}{\otimes} A' \underset{\min}{\otimes} H$$

restricts and corestricts an equivariant *-homomorphism $f : A \circledast^{\delta} H \to A' \circledast^{\delta'} H$. Hence, by the Pulling Back Theorem, the associated module $(A \circledast^{\delta} H) \Box V$ is not free, and the lemma follows. $\Box$

Iterated equivariant joins of compact quantum

Let (H, Δ) be a compact quantum group such that the C*-algebra H admits a character χ. Then

$$\mathrm{ev}_{\frac{1}{2}} \otimes \chi \otimes \mathsf{id} : H \circledast^{\Delta} H \longrightarrow H$$

is an equivariant *-homomorphism. Now, by the preceding lemma, if V is a finite-dimensional representation of (H, Δ) such that the module $(H \circledast^{\Delta} H)\Box V$ is not free, then there does not exist an equivariant *-homomorphism $H \to (H \circledast^{\Delta} H) \circledast^{\mathsf{id}\otimes\Delta} H$. By induction, one can extend this conclusion to an arbitrary finitely iterated equivariant join of H with itself.

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Furthermore, one can prove that for any finite-dimensional representation V of a compact quantum group (H, Δ), the associated finitely-generated projective module $(H \circledast^{\Delta} H) \Box_H V$ is represented by a Milnor idempotent $p_{U^{-1}}$, where U is a matrix of the representation V.

Quantum quaternionic projective spaces

Consider the defining fibration of the quaternionic projective space:

$$SU(2) * \cdots * SU(2) \cong S^{4n+3}, \quad S^{4n+3}/SU(2) = \mathbb{H}P^n.$$

To obtain a q-deformation of this fibration, we take $H = C(SU_q(2))$ and A equal to a finitely iterated equivariant join of H. The quantum principal $SU_q(2)$-bundle thus given is *not* trivializable:

Theorem

*There does not exist a $C(SU_q(2))$-equivariant *-homomorphism $f\colon C(SU_q(2)) \to A \circledast^{\delta} C(SU_q(2))$ (Borsuk-Ulam).*

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Proof outline: If V is the fundamental representation of $SU_q(2)$, then index paring considerations applied to the associated Milnor idempotent $p_{U^{-1}}$ show that

$$C(SU_q(2)) \circledast^{\Delta} C(SU_q(2))) \Box_{C(SU_q(2))} V$$

is not stably free. Thus, as $C(SU_q(2))$ admits characters, the assumptions of the Key Lemma are satisfied, whence f does not exist. $\square$

Quantum Dynamics, 2016–2019

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