



**THERE AND BACK AGAIN:
FROM THE BORSUK-ULAM THEOREM
TO QUANTUM SPACES**

Piotr M. Hajac (IMPAN / University of New Brunswick)
Tatiana Shulman (IMPAN)

Joint work with
Paul F. Baum, Ludwik Dąbrowski and Tomasz Maszczyk

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- It is therein that the generalized Borsuk-Ulam-type statements dwell waiting to be proven or disproven.
- We end by paying tribute to the ancient quantum group $SU_q(2)$, and showing the non-trivializability of the $SU_q(2)$ compact quantum principal bundle $S_q^{4n+3} \rightarrow \mathbb{H}P_q^n$ defining noncommutative quaternionic projective spaces. This is the main result, which is a special case of the type II noncommutative Borsuk-Ulam conjecture.

Jiří Matoušek

Using the Borsuk-Ulam Theorem

Lectures on Topological Methods
in Combinatorics and Geometry



The Borsuk-Ulam Theorem

Theorem (Borsuk-Ulam)

Let n be a positive natural number. If $f: S^n \rightarrow \mathbb{R}^n$ is continuous, then there exists a pair $(p, -p)$ of antipodal points on S^n such that $f(p) = f(-p)$.

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The logical negation of the theorem

There exists a continuous map $f: S^n \rightarrow \mathbb{R}^n$ such that for all pairs $(p, -p)$ of antipodal points on S^n we have $f(p) \neq f(-p)$.

The Borsuk-Ulam Theorem reformulated

For the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on S^n and \mathbb{R}^n , the latter statement is equivalent to:

Equivalent negation

There exists a $\mathbb{Z}/2\mathbb{Z}$ -equivariant continuous map $\tilde{f}: S^n \rightarrow S^{n-1}$.

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Indeed, if $f: S^n \rightarrow \mathbb{R}^n$ is a continuous map with $f(p) \neq f(-p)$, then the formula

$$\tilde{f}(p) := \frac{f(p) - f(-p)}{\|f(p) - f(-p)\|}$$

defines a continuous $\mathbb{Z}/2\mathbb{Z}$ -equivariant map from S^n to S^{n-1} .

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Also, composing any such a map with the inclusion map $S^{n-1} \subset \mathbb{R}^n$ yields a nowhere vanishing continuous map $f: S^n \rightarrow \mathbb{R}^n$ with $f(-p) = -f(p) \neq f(p)$.

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Theorem (equivariant formulation)

*Let n be a positive natural number. There does **not** exist a $\mathbb{Z}/2\mathbb{Z}$ -equivariant continuous map $\tilde{f}: S^n \rightarrow S^{n-1}$.*

Famous corollaries

Theorem (The Brouwer Fixed Point Theorem)

*Let n be any positive integer, and B^n be a ball of dimension n .
Then every continuous map $f : B^n \rightarrow B^n$ possesses a fixed point.*

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Theorem (The sandwich theorem)

Let n be any positive integer. Given n measurable “objects” in the n -dimensional Euclidean space, it is possible to divide all of them in half (with respect to their measure, i.e. volume) with a single $(n - 1)$ -dimensional hyperplane.

What is a compact quantum space?

Definition (concrete)

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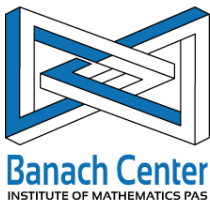
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Copernican-style revolution

Given a compact Hausdorff space of points, we can define the C*-algebra of functions on the space, but the central concept is that of a commutative C*-algebras, and points appear as characters (algebra homomorphisms into \mathbb{C}) rather than as primary objects. We think of noncommutative unital C*-algebras as algebras of functions on **compact quantum spaces**.

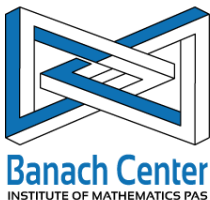


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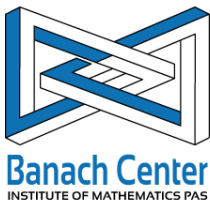
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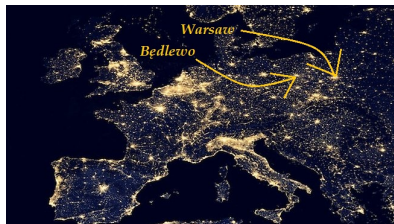
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**GEOMETRY, REPRESENTATION THEORY
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Sponsored by:

- The Fields Institute, University of Toronto, Canada
- National Science Foundation, USA
- The Pennsylvania State University, USA



FIELDS



What is a compact quantum group?

Definition (S. L. Woronowicz)

A **compact quantum group** is a unital C^* -algebra H with a given unital $*$ -homomorphism $\Delta: H \rightarrow H \otimes_{\min} H$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes_{\min} H \\ \downarrow \Delta & & \downarrow \Delta \otimes \text{id} \\ H \otimes_{\min} H & \xrightarrow{\text{id} \otimes \Delta} & H \otimes_{\min} H \otimes_{\min} H \end{array}$$

commutes and the two-sided cancellation property holds:

$$\{(a \otimes 1)\Delta(b) \mid a, b \in H\}^{\text{cls}} = H \otimes_{\min} H = \{\Delta(a)(1 \otimes b) \mid a, b \in H\}^{\text{cls}}.$$

Here “cls” stands for “closed linear span”.

Free actions of compact quantum groups

Let A be a unital C^* -algebra and $\delta : A \rightarrow A \otimes_{\min} H$ a unital $*$ -homomorphism. We call δ a **coaction** of H on A (or an action of the compact quantum group (H, Δ) on A) iff

- 1 $(\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \Delta) \circ \delta$ (coassociativity),
- 2 $\{\delta(a)(1 \otimes h) \mid a \in A, h \in H\}^{\text{cls}} = A \otimes_{\min} H$ (counitality)
- 3 $\ker \delta = 0$ (injectivity).

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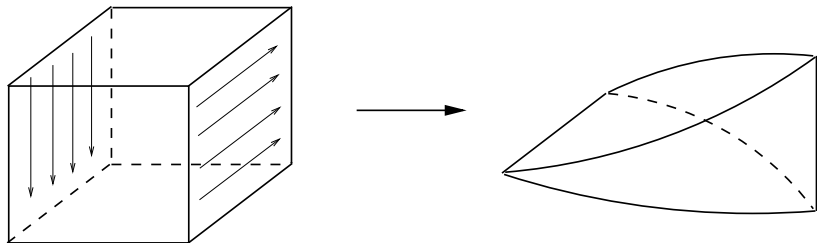
Definition (D. A. Ellwood)

A coaction δ is called **free** iff

$$\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\text{cls}} = A \otimes_{\min} H .$$

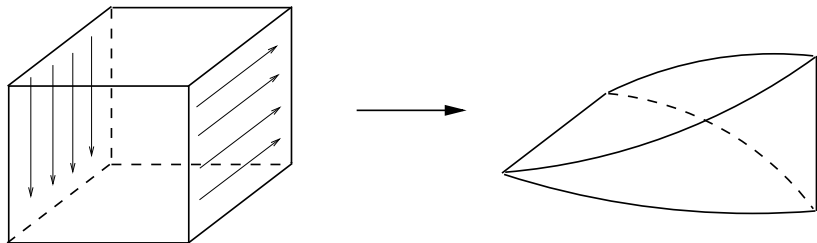
Equivariant join construction

For any topological spaces X and Y , one defines the **join** space $X * Y$ as the quotient of $[0, 1] \times X \times Y$ by a certain equivalence relation:



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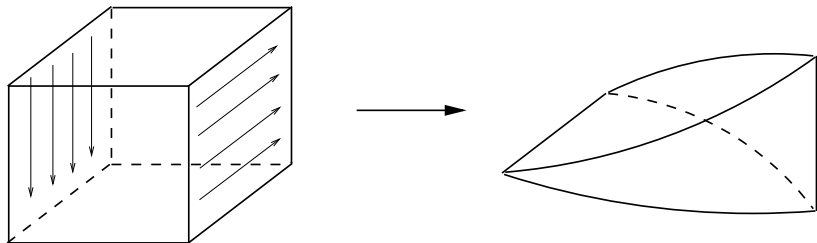
For any topological spaces X and Y , one defines the **join** space $X * Y$ as the quotient of $[0, 1] \times X \times Y$ by a certain equivalence relation:



If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G , then the diagonal action of G on the join $X * G$ is again continuous and free.

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If X is a compact Hausdorff space with a continuous free action of a compact Hausdorff group G , then the diagonal action of G on the join $X * G$ is again continuous and free. In particular, for the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on S^{n-1} , we obtain a $\mathbb{Z}/2\mathbb{Z}$ -equivariant identification $S^n \cong S^{n-1} * \mathbb{Z}/2\mathbb{Z}$ for the antipodal and diagonal actions respectively.

Gauged equivariant join construction

If $Y = G$, we can construct the join G -space $X * Y$ in a different manner: at 0 we collapse $X \times G$ to G as before, and at 1 we collapse $X \times G$ to $(X \times G)/R_D$ instead of X . Here R_D is the equivalence relation generated by

$$\boxed{(x, h) \sim (x', h'), \text{ where } xh = x'h'}.$$

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More precisely, let R'_J be the equivalence relation on $I \times X \times G$ generated by

$$(0, x, h) \sim (0, x', h) \quad \text{and} \quad (1, x, h) \sim (1, x', h'), \text{ where } xh = x'h'.$$

The formula $[(t, x, h)]k := [(t, x, hk)]$ defines a continuous right G -action on $(I \times X \times G)/R'_J$, and the formula

$$X * G \ni [(t, x, h)] \longmapsto [(t, xh^{-1}, h)] \in (I \times X \times G)/R'_J$$

yields a G -equivariant homeomorphism.

Join formulation and classical generalization

Thus the Borsuk-Ulam Theorem is equivalent to:

Theorem (join formulation)

*Let n be a positive natural number. There does **not** exist a $\mathbb{Z}/2\mathbb{Z}$ -equivariant continuous map $\tilde{f}: S^{n-1} * \mathbb{Z}/2\mathbb{Z} \rightarrow S^{n-1}$.*

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This naturally leads to:

A classical Borsuk-Ulam-type conjecture

Let X be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group G . Then, for the diagonal action of G on $X * G$, there does **not** exist a G -equivariant continuous map $f: X * G \rightarrow X$.

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Corollary

Ageev's conjecture about the Menger compacta.

Equivariant noncommutative join construction

Definition (L. Dąbrowski, T. Hadfield, P. M. H.)

For any compact quantum group (H, Δ) acting freely on a unital C^* -algebra A , we define its **equivariant join** with H to be the unital C^* -algebra

$$A \overset{\delta}{\circledast} H := \left\{ f \in C([0, 1], A) \underset{\min}{\otimes} H \mid f(0) \in \mathbb{C} \otimes H, f(1) \in \delta(A) \right\}.$$

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Theorem (P. F. Baum, K. De Commer, P. M. H.)

The \ast -homomorphism

$$\text{id} \otimes \Delta: C([0, 1], A) \underset{\min}{\otimes} H \longrightarrow C([0, 1], A) \underset{\min}{\otimes} H \underset{\min}{\otimes} H$$

defines a free action of the compact quantum group (H, Δ) on the equivariant join C^ -algebra $A \overset{\delta}{\ast} H$.*

Noncommutative Borsuk-Ulam-type conjectures

Conjecture 1

Let A be a unital C^* -algebra with a free action of a non-trivial compact quantum group (H, Δ) . Then there **does not exist an H -equivariant $*$ -homomorphism $A \rightarrow A \otimes^{\delta} H$.**

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The classical case

If X is a compact Hausdorff principal G -bundle, $A = C(X)$ and $H = C(G)$, then Conjecture 2 states that the principal G -bundle $X * G$ is not trivializable unless G is trivial. This is clearly true because otherwise **$G * G$ would be trivializable, which is tantamount to G being contractible**, and the only contractible compact Hausdorff group is trivial.

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Iterated joins of the quantum $SU(2)$ group

Consider the defining fibration of the quaternionic projective space:

$$SU(2) * \cdots * SU(2) \cong S^{4n+3}, \quad S^{4n+3}/SU(2) = \mathbb{H}P^n.$$

To obtain a q -deformation of this fibration, we take $H = C(SU_q(2))$ and A equal to a finitely iterated equivariant join of H . The quantum principal $SU_q(2)$ -bundle thus given is *not* trivializable:

Theorem (main)

There does **not** exist a $C(SU_q(2))$ -equivariant $*$ -homomorphism $f: C(SU_q(2)) \rightarrow A \otimes^\delta C(SU_q(2))$.

Iterated joins of the quantum $SU(2)$ group

Consider the defining fibration of the quaternionic projective space:

$$SU(2) * \cdots * SU(2) \cong S^{4n+3}, \quad S^{4n+3}/SU(2) = \mathbb{H}P^n.$$

To obtain a q -deformation of this fibration, we take $H = C(SU_q(2))$ and A equal to a finitely iterated equivariant join of H . The quantum principal $SU_q(2)$ -bundle thus given is *not* trivializable:

Theorem (main)

There does *not* exist a $C(SU_q(2))$ -equivariant $*$ -homomorphism $f: C(SU_q(2)) \rightarrow A \otimes^\delta C(SU_q(2))$.

Proof outline: First, we prove that for any finite-dimensional representation V of a compact quantum group (H, Δ) , the associated finitely-generated projective module $(H \otimes^\Delta H) \square_H V$ is represented by a Milnor idempotent $p_{U^{-1}}$, where U is a matrix of the representation V . Then we choose V to be the fundamental representation of $SU_q(2)$, and infer from index pairing considerations that $(H \otimes^\Delta H) \square_H V$ is not stably free. Finally, by the Pulling Back Theorem, we conclude that $(A \otimes^\delta H) \square_H V$ is *not* stably free, whence f does not exist. \square

- *Noncommutative Borsuk-Ulam-type conjectures*;
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Paul F. Baum, Ludwik Dąbrowski, Piotr M. Hajac;
Banach Center Publications 106 (2015), 9–18.
- *Pulling back noncommutative associated vector bundles*;
Piotr M. Hajac, Tomasz Maszczyk;
arXiv:1601.00021.

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HORIZON 2020

The EU Framework Programme for Research and Innovation