NONCOMMUTATIVE BORSUK-ULAM-TYPE CONJECTURES REVISITED

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Borsuk-Ulam Theorem (1933):

$\forall n \in \mathbb{N}\ \nexists\ \mathbb{Z}/2\mathbb{Z}$-equiv. cont. $S^{n+1} \to S^n$

Join formulation of the BUT:

$$\nexists\ S^n * \mathbb{Z}/2\mathbb{Z} \to S^n$$

Join generalization (2015):

Let X be a compact Hausdorff space equipped with a continuous free action of a (non-trivial) compact Hausdorff group G. Then:

$\forall n \in \mathbb{N}\ \nexists\ G$-equiv. cont. $X * G \to X$.

Corollary: $\nexists\ G$-equiv. cont. $X * G \to G$, i.e. $X * G$ is not trivializable.

Indeed, by choosing any $x_0 \in X$, we get

$$X * G \longrightarrow G \xrightarrow{x_0} X \quad (g \mapsto x_0 g).$$

Next, by taking $X = G$, we conclude that the only contractible compact Hausdorff group is the trivial group.

<u>Theorem</u>: Let G be a non-trivial compact connected semisimple Lie group acting freely and continuously on a compact Hausdorff space X. Then the exists a finite-dim. rep. $\rho : G \to GL(V)$ s.t. the ass. vec. bundle $(X * G) \times_\rho V \longrightarrow (X * G)/G$ is <u>not</u> stably trivial.

<u>Remark</u>: The connectedness assumption for G is necessary because for $X = G = \mathbb{Z}/2\mathbb{Z}$ we have $(X * G)/G = (G * G)/G$ $= \Sigma G = \Sigma\, \mathbb{Z}/2\mathbb{Z} = S^1$, and $K^0(S^1) = \mathbb{Z}$

Proof outline: Note first that, if $f : Y' \to Y$ is a cont. map intertwining cont. free actions of G, then

$$Y' \cong Y'/G \underset{Y/G}{\times} Y, \qquad p \mapsto ([p], f(p)),$$

as G-spaces. Hence $X' \times_{g} V = X'/G \underset{X/G}{\times} (X \times_{g} V)$ $= f^{*}(X \times_{g} V)$, so that stable triviality of $X \times_{g} V$ implies stable triviality of $X' \times_{g} V$.

Now, since $G \ni g \mapsto x_0 g \in X$ induces a G-equivariant cont. map $G * G \to X * G$, by the above argument, it suffices to find $g : G \to GL(V)$ s.t. $(G * G) \times_{g} V$ is not stably trivial.

To do so, we take a non-trivial $u \in K^{1}(G)$, note that $\Sigma G = CG \underset{G}{\amalg} CG$, apply the Milnor connecting homomorphism to get a non-trivial element of $K^0(\Sigma G)$, and identify it with $(G*G)\times_\xi V$ for a suitable ξ depending on n. $\square$

Enter the NCG !

① Take a <u>free</u> action of a compact quantum group (H,Δ) on a unital C^*-algebra A.

② Define a noncommutative join C^*-algebra $A \circledast H$ equipped with a free action of (H,Δ).

③ Take a f.-dim. rep. V of (H,Δ) and, using Peter-Weyl theory, construct an associated finitely generated projective module

$$(A \circledast H) \,\Box\, V$$ over the fixed-point subalgebra $(A \circledast H)^{(H,\Delta)}$.

<u>Theorem</u>: Let (H, Δ) be a q-deformation of a compact connected semisimple Lie group. Let A be any unital C^*-alg. admitting a character and equipped with a free action of (H,Δ). Then there exists a f.-dim. rep. V of (H,Δ) s.t.

$$[(A \circledast H) \,\square\, V] \notin \mathbb{Z}[1] \subseteq K_0\left((A \circledast H)^{(H,\Delta)}\right).$$