Market Modelling of CDOs

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Introduction

Essentials of securitization

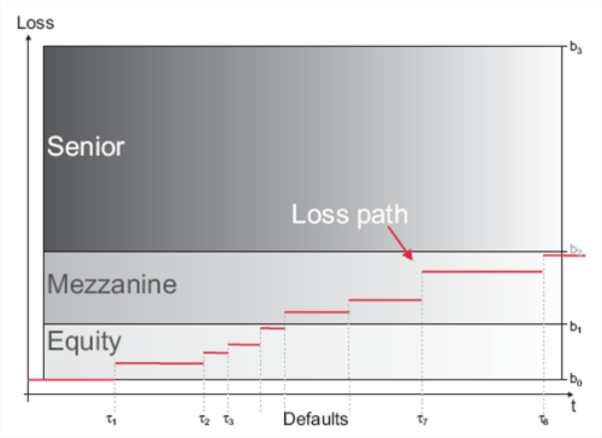
- Consider a CDO as a pool of m defaultable entities.
- Default i occurs at τ_i with associated loss q_i
- Essential process: cumulative loss

$$L_t = \sum_{i=1}^{m} q_i 1_{\{\tau_i \leq t\}}.$$

- Normalize the total nominal to 1, set $\mathcal{I} := [0, 1]$.
- Loss is split into tranches: a tranche refers to an interval $(x_i, x_{i-1}] \subset \mathcal{I}$,

$$0 = x_0 < x_1 < \cdots < x_k = 1$$

Partition of losses into tranches



Single tranche CDOs

A STCDO is specified by

- a number of future dates $T_0 < T_1 < \cdots < T_m$,
- a *tranche* with lower and upper detachment points $x_1 < x_2$,
- a fixed spread S.

We write

$$H(x) := (x_2 - x)^+ - (x_1 - x)^+ = \int_{(x_1, x_2]} 1_{\{x \leq y\}} dy.$$

An investor in this STCDO

- receives $SH(L_{T_k})$ at T_k, $k = 1, \ldots, m-1$ (payment leg),
- pays $H(L_{T_{k+1}}) - H(L_{T_k})$ at any T_{k+1},$k = 1, \ldots, m-1$. (default leg)

- A security which pays $1_{\{L_T \leq x\}}$ at T is called (T,x)-*bond*. Its price at time $t \leq T$ is denoted by $P(t,T,x)$.

- If the market is free of arbitrage, $P(t,T,x)$ is nondecreasing in x and $P(t,T,1) = P(t,T)$. (risk-free bond)

Let us denote by $e(t,T_{k+1},x)$ the value at time t of the payment given by $\mathbf{1}_{\{A_{T_k} \leq x, A_{T_{k+1}} > x\}}$ at the tenor date T_{k+1}.

Proposition

The value of the STCDO at time $t \leq T_1$ is

$$\pi^{STCDO}(t,S) = \int_{x_1}^{x_2} \left(S \sum_{k=1}^{m-1} P(t,T_k,y) - \sum_{k=1}^{m-1} e(t,T_{k+1},y) \right) dy. \tag{1}$$

Solving $\pi^{STCDO} = 0$ for S gives the fair spread.

Dynamic forward-rate models for CDO markets

Drift condition

- $A_t = \sum_{s \le t} \Delta A_s$ is an increasing marked point process with compensator $\nu^A(t,dx)\,dt$ and values in $[0,1]$.
- Consider $\lambda(t,x)$, such that

$$M_t^x = 1_{\{A_t \le x\}} + \int_0^t 1_{\{A_s \le x\}} \lambda(s,x)\,ds$$

  is a martingale.
- Consider a d-dimensional Lévy process X such that $\mathbb{E}(\,e^{-\langle u, X_t \rangle}) = e^{tJ(u)}$ $u \in \mathbb{R}^d$ with

$$J(u) = \langle m, u \rangle + \frac{1}{2}\langle \Sigma u, u \rangle + \int_{\mathbb{R}^d} \Big(e^{-\langle u,z \rangle} - 1 + 1_{\{|z| \le 1\}}(z)\,\langle u, z \rangle \Big) \tilde{\nu}(dz).$$

HJM-like approach:

We consider

$$P(t,T,x) = 1_{\{A_t \le x\}} \exp\Big(-\int_t^T f(t,u,x)\,du\Big)$$

where

$$\begin{aligned} f(t,T,x) = f(0,T,x) + \int_0^t a(s,T,x)ds + \int_0^t \langle b(s,T,x), dZ_s\rangle \\ + \int_0^t \int_{\mathcal{I}} c(s,T,x;y)\,\mu^L(ds,dy) \end{aligned} \tag{2}$$

No-arbitrage condition

$$\left(e^{-\int_0^t r_s ds} P(t, T, x)\right)_{0 \leq t \leq T} \text{ are local martingales for all } (T, x). \tag{3}$$

Under some technical assumptions we have:

Theorem

(3) *is equivalent to*

$$\int_t^s a(t, u, x) du = J\left(\int_t^s b(t, u, x) du\right) \tag{4}$$

$$f(t, t, x) = f(t, t, 1) + \lambda(t, x), \tag{5}$$

on $\{L_t \leq x\}$, *a.s.*

Libor market models

The (T_k, x)-Libor rate

Denote $\mathcal{T} := \{T_0, \ldots, T_n\}$, $\delta_k := T_{k+1} - T_k$ and let

$$P(t, T, x) = p(t, T, x) 1_{\{A_t \leq x\}}, \tag{6}$$

$(p(t, T, x))_{0 \leq t \leq T}$ a strictly positive special semimartingale with $p(T, T, x) = 1$.

Definition

The (T_k, x)-*Libor rate* at time $t \leq T_k$ is given by

$$L(t, T_k, x) := \mathbf{1}_{\{A_t \leq x\}} \frac{1}{\delta_k} \left(\frac{p(t, T_k, x)}{p(t, T_{k+1}, x)} - 1 \right); \tag{7}$$

the (T_k, x)-*credit spread* is defined by

$$H(t, T_k, x) := \mathbf{1}_{\{A_t \leq x\}} \frac{L(t, T_k, x) - L(t, T_k)}{1 + \delta_k L(t, T_k)}. \tag{8}$$

The (T_k, x)-*forward price* is given by

$$F(t, T_k, x) := \frac{P(t, T_k, x)}{P(t, T_k)}. \tag{9}$$

The quantities $H(t, T_k, x)$ represent the discrete-tenor analogs of credit spreads in the continuous case: under

$$P(t, T_k, x) = \mathbf{1}_{\{A_t \le x\}} e^{-\int_t^{T_k} f(t,u,x)du},$$

by the definition of the Libor rate

$$\begin{aligned} L(t, T_k) &= \frac{1}{\delta_k}\left(e^{\int_{T_k}^{T_{k+1}} f(t,u,1)du} - 1\right) \\ &\approx \frac{1}{\delta_k}\int_{T_k}^{T_{k+1}} f(t,u,1)du. \end{aligned}$$

This shows that the Libor rate is approximately the average forward rate over $[T_k, T_{k+1}]$.

Furthermore, on $A_t \leq x$,

$$
\begin{aligned}
H(t, T_k, x) &= \frac{L(t, T_k, x) - L(t, T_k)}{1 + \delta_k L(t, T_k)} \\
&= \frac{1}{\delta_k} \left(e^{-\int_{T_k}^{T_{k+1}} (f(t,u,x) - f(t,u,1)) du} - 1 \right) \\
&\approx \frac{1}{\delta_k} \int_{T_k}^{T_{k+1}} (f(t, u, x) - f(t, u, 1)) du,
\end{aligned}
$$

which is approximately the average forward credit spread over the time interval $[T_k, T_{k+1}]$.

Relation between STCDO and Libor

How to extract the Libor rates from observed STCDO prices? Recal (1) and assume that

$$P(t, T_k, x) = P(t, T_k)\mathbb{E}_{\mathbb{Q}_{T_{k+1}}}(\mathbf{1}_{\{A_{T_k} \le x\}}|\mathcal{G}_t), \tag{10}$$

for all $(T_k, x) \in \mathcal{T} \times \mathcal{I}$ and $0 \le t \le T_k$. Note that (10) is equivalent to

$$\mathbb{E}_{\mathbb{Q}_{T_{k+1}}}(\mathbf{1}_{\{A_{T_k} \le x\}}|\mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}_{T_k}}(\mathbf{1}_{\{A_{T_k} \le x\}}|\mathcal{G}_t).$$

Lemma

Assume (10)*. Then*

$$e(t, T_{k+1}, x) = \frac{P(t, T_{k+1})}{P(t, T_k)} P(t, T_k, x) - P(t, T_{k+1}, x)$$

for $k = 1, \dots, m-1$ *and* $x \in \mathcal{I}$*.*

Assume now that risk-free Libor rates and STCDO prices at time t are observed for maturities $T_1, \ldots, T_m$ and levels (x_{i-1}, x_i) with $i = 1, \ldots, n$ where $x_0 = 0$ and $x_n = 1$ and (10) holds.

- **Step 1** For maturity T_1 the default leg equals

$$\int_{x_{i-1}}^{x_i} \big(P(t, T_1) - P(t, T_1, y)\big) dy.$$

This allows directly to compute

$$P(t, T_1, x_{i-1}, x_i) := \int_{x_{i-1}}^{x_i} P(t, T_1, y) dy$$

for all $i = 1, \ldots, n$.

- **j → j+1** Assume that the values $P(t, T_k, x_{i-1}, x_i)$ are given for all $i = 1, \dots, n$ and $k = 1, \dots, j$. A STCDO with maturity T_{j+1} satisfies

$$S(t, T_{j+1}, x_{i-1}, x_i) = \frac{\sum_{k=1}^{j} \left(\frac{P(t,T_{k+1})}{P(t,T_k)} P(t, T_k, x_{i-1}, x_i) - P(t, T_{k+1}, x_{i-1}, x_i) \right)}{\sum_{k=1}^{j} P(t, T_k, x_{i-1}, x_i)}.$$

The denominator is given as a sum of quantities which have been computed in the previous j steps. The numerator equals

$$\frac{P(t, T_{j+1})}{P(t, T_j)} P(t, T_j, x_{i-1}, x_i) - P(t, T_{j+1}, x_{i-1}, x_i) + \sum_{k=1}^{j-1} \left(\frac{P(t, T_{k+1})}{P(t, T_k)} P(t, T_k, x_{i-1}, x_i) - P(t, T_{k+1}, x_{i-1}, x_i) \right)$$

Therefore $P(t, T_{j+1}, x_{i-1}, x_i)$ equals

$$\begin{aligned} &\frac{P(t, T_{j+1})}{P(t, T_j)} P(t, T_j, x_{i-1}, x_i) \\ &+ \sum_{k=1}^{j-1} \left(\frac{P(t, T_{k+1})}{P(t, T_k)} P(t, T_k, x_{i-1}, x_i) - P(t, T_{k+1}, x_{i-1}, x_i) \right) \\ &- S(t, T_{j+1}, x_{i-1}, x_i) \sum_{k=1}^{j} P(t, T_k, x_{i-1}, x_i) \end{aligned} \tag{11}$$

and this step is completed.
In this way one is able to extract (T_k, x)-rates from STCDO prices.

If we assume in addition zero risk-free interest rates, we obtain the following formula for the default leg of the STCDO:

$$\sum_{k=1}^{m-1} \int_{x_{i-1}}^{x_i} e(t, T_{k+1}, y) dy = \int_{x_{i-1}}^{x_i} \big(P(t, T_1, y) - P(t, T_m, y)\big) dy.$$

Moreover, in the above algorithm (11) simplifies to

$$P(t, T_{j+1}, x_{i-1}, x_i) = P(t, T_1, x_{i-1}, x_i) \\ - S(t, T_{j+1}, x_{i-1}, x_i) \sum_{k=1}^{j} P(t, T_k, x_{i-1}, x_i).$$

Recall

$$H(t, T_k, x) := \mathbf{1}_{\{A_t \le x\}} \frac{L(t, T_k, x) - L(t, T_k)}{1 + \delta_k L(t, T_k)}$$

and let

$$h(t, T_k, x) := \frac{L(t, T_k, x) - L(t, T_k)}{1 + \delta_k L(t, T_k)}.$$

Lemma

Assume (**A9**). *Let* $x \in \mathcal{I}$ *and* $k \in \{1, \dots, m-1\}$. *Then, for every* $t \le T_k$,

$$e(t, T_{k+1}, x) = \delta_k P(t, T_{k+1}, x) \mathbb{E}_{\mathbb{Q}_{T_{k+1},x}}\big(h(T_k, T_k, x) | \mathcal{G}_t\big).$$

Consider a complete stochastic basis $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q}_{T^*})$
We assume that $A_t = \sum_{s \leq t} \Delta A_s$ is an $\mathcal{I}$-valued increasing marked point process with absolutely continuous $\mathbb{Q}^*$-compensator

$$\nu^A(dt, dy) = F_t^A(dy)dt, \tag{12}$$

where F^A is a transition kernel from $(\Omega \times [0, T^*], \mathcal{P})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\mathcal{P}$ denotes the predictable σ-algebra on $\Omega \times [0, T^*]$. Let $\lambda(t, x) = \nu^A(t, (x - A_t, 1] \cap \mathcal{I})$; such that

$$M_t^x = \mathbf{1}_{\{A_t \leq x\}} + \int_0^t \mathbf{1}_{\{A_s \leq x\}} \lambda(s, x) ds \tag{13}$$

is a $\mathbb{Q}^*$-martingale

Let

$$X = (X^1, \dots, X^d, X^{d+1}) = (\tilde{X}, X^{d+1})$$

be an $\mathbb{R}^{d+1}$-valued special semimartingale on the stochastic basis $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q}^*)$ with $X_0 = 0$. Assume that $\tilde{X}$ is a time-inhomogeneous Lévy process and X^{d+1} is a pure-jump process with compensator

$$\nu^A(dt, dy) = F^A(dy)dt.$$

Recall $H(t, T_k, x) = 1_{\{A_t \le x\}} h(t, T_k, x)$ and then

$$1 + \delta_k L(t, T_k, x) = (1 + \delta_k L(t, T_k))(1 + \delta_k h(t, T_k, x))$$

such that

$$L(t, T_k, x) = \mathbf{1}_{\{A_t \le x\}} \frac{1}{\delta_k} \Big((1 + \delta_k L(t, T_k))(1 + \delta_k h(t, T_k, x)) - 1 \Big). \tag{14}$$

In other words, every forward (T_k, x)-Libor rate can be obtained from the risk-free forward Libor rate with the same maturity and the corresponding pre-default credit spread. Note that

$$L(t, T_k, x) > L(t, T_k) \iff h(t, T_k, x) > 0$$

on $\{A_t \le x\}$. Hence, $h(\cdot, T_k, x) > 0$ ensures that the defaultable forward (T_k, x)-Libor rates are higher than their risk-free counterparts, an important property in practice.

Denote

$$\widetilde{\gamma}(s, T_k, x) := (\gamma^1(s, T_k, x), \dots, \gamma^d(s, T_k, x)).$$

We assume that the pre-default credit spread h follows:

(A7) For every $t \le T_k$

$$h(t, T_k, x) = h(0, T_k, x) \exp\left(\int_0^t b(s, T_k, x) ds + \int_0^t \widetilde{\gamma}(s, T_k, x) d\widetilde{X}_s^{T_{k+1}} + \int_0^t \int_{\mathcal{I}} c(s, T_k, x; y)(\mu^A - \nu^{A, T_{k+1}})(ds, dy) \right)$$

with the initial condition

$$h(0, T_k, x) = \frac{1}{\delta_k} \left(\frac{F(0, T_k, x)}{F(0, T_{k+1}, x)} - 1 \right).$$

The drift term $b(\cdot, T_k, \cdot)$ is an $\mathbb{R}$-valued process with $b(s, T_k, x) = 0$.

(A4) For all T_k there is a deterministic, $\mathbb{R}_+^{d+1}$-valued function $\gamma(s, T_k, x)$, which as a function of $(s, x) \mapsto \gamma(s, T_k, x)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{I})$-measurable. Moreover,

$$\gamma^{d+1}(s, T_k, x) = 0 \quad \text{and} \quad \sum_{k=1}^{n-1} (\sigma^j(s, T_k) + \gamma^j(s, T_k, x)) \leq C,$$

for all $s \in [0, T^*]$ and every coordinate $j \in \{1, \dots, d+1\}$, where C is the constant from (**A1**). If $s > T_k$, then $\gamma(s, T_k, x) = 0$.

(A5) For all T_k there is an $\mathbb{R}$-valued function $c(s, T_k, x; y)$, which is called the *contagion* parameter and which as a function of $(s, x, y) \mapsto c(s, T_k, x; y)$ is $\mathcal{P} \otimes \mathcal{B}(\mathcal{I}) \otimes \mathcal{B}(\mathcal{I})$-measurable. We also assume

$$\sup_{s \leq T_k, x, y \in \mathcal{I}, \omega \in \Omega} |c(s, T_k, x; y)| < \infty$$

and $c(s, T_k, x; y) = 0$ for $s > T_k$.

(A6) The initial term structure $P(0, T_k, x)$ is strictly positive, strictly decreasing in k and satisfies

$$F(0, T_k, x) = \frac{P(0, T_k, x)}{P(0, T_k)} \geq \frac{P(0, T_{k+1}, x)}{P(0, T_{k+1})} = F(0, T_{k+1}, x).$$

Absence of Arbitrage in a Market Model

By absence of arbitrage we mean that for each $i, k = 1, \dots, n$ the bond price process

$$\Big(\frac{P(t, T_k, x)}{P(t, T_i)}\Big)_{0 \leq t \leq T_i \wedge T_k}$$

is a local martingale with respect to the corresponding forward measure $\mathbb{Q}_{T_i}$. If the risk-free market is free of arbitrage, then this is equivalent to the following:

For each $k = 1, \dots, n$ the process

$$\Big(\frac{P(t, T_k, x)}{P(t, T_k)}\Big)_{0 \leq t \leq T_k}$$

is a $\mathbb{Q}_{T_k}$-local martingale.

Let us denote for each $t \leq T_k,\ k = 1, \ldots, n-1$, and $x \in \mathcal{I}$

$$y(t, T_k, x) := \frac{1}{1 + \delta_k h(t, T_k, x)}. \tag{15}$$

In the following lemma we deduce the connection between the forward (T_k, x)-bond price processes and y.

Lemma

Consider $t \in (0, T_{k-1}]$, where $t \in (T_{l-1}, T_l]$ for some $l \in \{1, \ldots, k-1\}$. Then

$$F(t, T_k, x) = \left(\prod_{i=l}^{k-1} y(t, T_i, x) \right) F(t, T_l, x). \tag{16}$$

The proof relies on the following relationship: On the set $\{A_t \leq x\}$,

$$1 + \delta_k H(t, T_k, x) = \frac{1 + \delta_k L(t, T_k, x)}{1 + \delta_k L(t, T_k)},$$

and hence

$$1 + \delta_k H(t, T_k, x) = \frac{p(t, T_k, x)}{P(t, T_k)} \left(\frac{p(t, T_{k+1}, x)}{P(t, T_{k+1})} \right)^{-1} > 0.$$

such that

$$H(t, T_k, x) = \frac{1}{\delta_k} \left(\frac{F(t, T_k, x)}{F(t, T_{k+1}, x)} - 1 \right).$$

Consequently, as soon as the pre-default intensities $h(\cdot, T_k, x)$ are specified, the forward (T_k, x)-bond price process is also partially specified. More precisely, according to the previous lemma, in order to describe completely the dynamics of $F(\cdot, T_k, x)$, it remains to specify for each $l = 1, \dots, n$ the dynamics of the process $F(\cdot, T_l, x)$ on the interval $(T_{l-1}, T_l]$. This can be done in different ways; the specification below being an obvious and simple choice.

(A8) For every $t \le T_k$ and $x \in \mathcal{I}$

$$\frac{p(t, T_k, x)}{P(t, T_k)} = \left(\prod_{i=0}^{k-1} y(t, T_i, x) \right) e^{\int_0^t b^P(s, T_k, x) ds},$$

where $b^P(\cdot, T_k, \cdot)$ is an $\mathbb{R}$-valued, locally integrable process. Recall that $y(t, T_i, x) = y(T_i, T_i, x)$, for $t \ge T_i$, by assumption (**A7**).

Then the forward (T_k, x)-bond price is given by

$$F(t, T_k, x) = \frac{p(t, T_k, x)}{P(t, T_k)} \mathbf{1}_{\{A_t \le x\}} = \left(\prod_{i=0}^{k-1} y(t, T_i, x) \right) e^{\int_0^t b^P(s, T_k, x) ds} \mathbf{1}_{\{A_t \le x\}}. \tag{17}$$

Remark

To ease notation we work with a continuous, finite variation process $e^{\int_0^t b^P(s,T_k,x)ds}$ in (**A8**). A more general specification with an exponential of some special semimartingale is possible and the occurring calculations can be done in the same way.

Before stating the main theorem of the section, which provides necessary and sufficient conditions for the forward (T_k, x)-bond price process (17) to be a $\mathbb{Q}_{T_k}$-local martingale, we need some auxiliary results.

Lemma

Assume (**A1**)–(**A7**′) *and let* $\tilde{Y}(\cdot, T_k, x) := \prod_{i=0}^{k-1} y(\cdot, T_i, x)$. *Then*

$$d\tilde{Y}(t, T_k, x) = \tilde{Y}(t-, T_k, x)\Bigg[D(t, T_k, x)dt - \sum_{i=1}^{k-1} g(t-, T_i, x)\sqrt{c_t}\gamma(t, T_i, x)dW_t^{T_k}$$

$$+ \int_{\mathbb{R}^{d+1}} \left(\prod_{i=1}^{k-1}\left(1 + g(t-, T_i, x)(e^{\varrho(t, T_i, x; y)} - 1)\right)^{-1} - 1\right)(\mu - \nu^{T_k})(dt, dy)\Bigg],$$

where

$$g(t, T_i, x) := \frac{\delta_i h(t, T_i, x)}{1 + \delta_i h(t, T_i, x)} \tag{18}$$

and ...

$$
\begin{aligned}
D(t, T_k, x) \quad := \quad & -\sum_{i=1}^{k-1} g(t-, T_i, x) b(t, T_i, x) \\
& + \sum_{i=1}^{k-1} g(t-, T_i, x) \Big\langle \gamma(t, T_i, x), c_t \sum_{j=i+1}^{k-1} \alpha(t, T_j) \Big\rangle \\
& - \sum_{i=1}^{k-1} \frac{1}{2} \big(g(t-, T_i, x) - g(t-, T_i, x)^2 \big) \| \sqrt{c_t} \gamma(t, T_i, x) \|^2 \\
& + \frac{1}{2} \Big\| \sum_{i=1}^{k-1} g(t-, T_i, x) \sqrt{c_t} \gamma(t, T_i, x) \Big\|^2 \qquad (19) \\
& + \int_{\mathbb{R}^{d+1}} \Bigg[\prod_{i=1}^{k-1} \Big(1 + g(t-, T_i, x)(e^{\varrho(t, T_i, x; y)} - 1) \Big)^{-1} - 1 \\
& + \sum_{i=1}^{k-1} g(t-, T_i, x) \varrho(t, T_i, x; y) \times \Bigg(\prod_{j=i+1}^{k-1} \beta(t, T_j, y) \Bigg) \Bigg] F_t^{T_k}(dy),
\end{aligned}
$$

Lemma

Assume (**A1**)–(**A8**). *The dynamics of the process* $\frac{p(\cdot, T_k, x)}{P(\cdot, T_k)}$ *under the forward measure* $\mathbb{Q}_{T_k}$ *is given by*

$$d\left(\frac{p(t, T_k, x)}{P(t, T_k)}\right) = \frac{p(t-, T_k, x)}{P(t-, T_k)}\Bigg(\Big(b^P(t, T_k, x) + D(t, T_k, x)\Big)dt$$

$$- \sum_{i=1}^{k-1} g(t-, T_i, x)\sqrt{c_t}\gamma(t, T_i, x)dW_t^{T_k}$$

$$+ \int_{\mathbb{R}^{d+1}} \left(\prod_{i=1}^{k-1}\Big(1 + g(t-, T_i, x)(e^{\varrho(t, T_i, x; y)} - 1)\Big)^{-1} - 1\right)$$

$$\times (\mu - \nu^{T_k})(dt, dy)\Bigg).$$

Theorem

Assume that **(A1)**–**(A8)** *are in force. Then the forward bond price process* $(F(t, T_k, x))_{0 \le t \le T_k}$ *is a* $\mathbb{Q}_{T_k}$*-local martingale if and only if*

$$D(t, T_k, x) = \lambda^{T_k}(t, x) - b^P(t, T_k, x) + \int_{\mathbb{R}^{d+1}} \left(\prod_{i=1}^{k-1} \Big(1 + g(t-, T_i, x)(e^{\varrho(t, T_i, x; y)} - 1)\Big)^{-1} - 1 \right) \times \mathbf{1}_{\{A_t + y^{d+1} > x\}} F_t^{T_k}(dy), \tag{20}$$

on the set $\{A_t \le x\}$*, for every* $t \in [0, T_k]$ $dt \times \mathbb{Q}_{T_k}$*-a.s.*

Example (Eberlein, Kluge, Schönbucher (2006))

This approach is a special case of our model in the doubly stochastic setting with no contagion, i.e. $c(\cdot, T_k; y) = 0$, for all T_k. Note that we can suppress x from the notation since in this case $\mathcal{I} = \{0\}$ and

$$\mathbf{1}_{\{A_t \leq 0\}} = \mathbf{1}_{\{\tau > t\}},$$

where τ is the default time of the considered defaultable bond.
Doubly stochastic means that the filtration $\mathbb{G}$ is given as $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$ is the *background filtration* (or the *reference filtration*) and the filtration $\mathbb{H} := (\mathcal{H}_t)$ is generated by the default time, i.e. $\mathcal{H}_t := \sigma(\mathbf{1}_{\{\tau \leq s\}}; 0 \leq s \leq t)$. Moreover, the default time τ is modeled as the first jump of the Cox process with hazard process denoted by Γ, i.e. Γ is an $\mathbb{F}$-adapted, right-continuous, increasing process such that $\Gamma_0 = 0$ and for every $t \leq T^*$

$$\mathbb{Q}_{T_n}(\tau > t | \mathcal{F}_t) = e^{-\Gamma_t}.$$

Example

Let us assume $\Gamma_t = \int_0^t \lambda_s ds$; λ remains the $\mathbb{F}$-intensity process of τ under *all* forward measures $\mathbb{Q}_{T_k}$. In EKS, the pre-default value $\overline{B}(\cdot, T_k)$ of the defaultable bond is specified as follows

$$\frac{\overline{B}(t, T_k)}{B(t, T_k)} := \prod_{i=0}^{k-1} \frac{1}{1 + \delta_i h(t, T_i)} e^{\Gamma_t}, \tag{21}$$

where $B(\cdot, T_k)$ is the default-free bond price process and where

$$h(t, T_k) \quad = \quad h(0, T_k) \exp\left(\int_0^t b^H(s, T_k) ds + \int_0^t \sqrt{c_s} \widetilde{\gamma}(s, T_k) d\widetilde{X}_s^{T_{k+1}} \right);$$

$\widetilde{X}^{T_{k+1}}$ being the d-dimensional special semimartingale obtained from the time-inhomogeneous Lévy process $\widetilde{X}$ by changing from $\mathbb{Q}_{T_n}$ to the forward measure $\mathbb{Q}_{T_{k+1}}$. By assumption, $\widetilde{X}$ is $\mathbb{F}$-adapted and so is $h(\cdot, T_k)$.

The no-arbitrage condition of EKS(2) is obtained as a special case of Theorem 7, stated in the following corollary.

Corollary

The forward defaultable bond price process $\frac{\overline{B}(\cdot,T_k)}{B(\cdot,T_k)}\mathbf{1}_{\{\tau>t\}}$ *with specification* (21) *is a* $(\mathbb{G},\mathbb{Q}_{T_k})$*-local martingale if and only if*

$$D(t, T_k) = 0,$$

for almost all $t \in [0, T_k]$*, or equivalently, if and only if the process*

$$\prod_{i=0}^{k-1} y(t, T_i) = \prod_{i=0}^{k-1} \frac{1}{1+\delta_i h(t, T_i)}, \qquad t \leq T_k,$$

is an $(\mathbb{F},\mathbb{Q}_{T_k})$*-local martingale.*

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