L^p-Theory of the linear Stokes Equation: Part I Function spaces, Stokes semigroup, Maximal L^p-Regularity

Matthias Hieber

TU Darmstadt

Department of Mathematics, Schlossgartenstr. 7, 64289 Darmstadt, Germany

Consider the flow of an incompressible, viscous fluid in a domain $\Omega \subset \mathbb{R}^n$ which is described by the equations of Navier-Stokes

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla \pi &= 0, \quad \text{in}\Omega \times (0,\infty), \\ \text{div} u &= 0, \quad \text{in } \Omega \times (0,\infty), \\ u(0) &= u_0, \quad \text{in } \Omega. \end{cases}$$

Here, u and π represent the velocity and pressure of the fluid, respectively.

In this talk we consider basic properties of the linear Stokes equation in the L^p-setting for $1 < p < \infty$ and later on also for the case $p = \infty$. Topics we discuss include

- Helmholtz projection and the space $L^p_\sigma(\Omega)$
- Resolvent estimates in half space $\mathbb{R}^n_+$
- The Stokes semigroup on $L^p(\Omega)$ for various type of domains $\Omega \subset \mathbb{R}^n$

References

[1] R. Denk, M. Hieber, J. Prüss, R-boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type, *Memoirs Amer. Math. Soc.*, vii + 114 pp., 2003.

[2] W. Desch, M. Hieber, J. Prüss, L^p-Theory of the Stokes operator in a half space *J. Evolution Eq.*, **1**, (2001), 115-142.

[3] M. Geissert, M. Hess, M. Hieber, C. Schwarz, K. Stavrakidis), Maximal $L^p - L^q$-estimates for the Stokes equation: a short proof of Solonnikov's theorem. *J. Math. Fluid Mechanics*, **12** (2010), 47-60.

[4] M. Geissert, H. Heck, M. Hieber, O. Sawada, Weak Neumann implies Stokes. *J. reine angew. Math.*, 669, (2012), 75-100.

L^p-Theory of the linear Stokes Equation: Part II H^∞-calculus, Domains with Non compact Boundaries

Matthias Hieber

TU Darmstadt

Department of Mathematics, Schlossgartenstr. 7, 64289 Darmstadt, Germany

In this talk we continue our investigation of the Stokes equation in various domains $\Omega \subset \mathbb{R}^n$, i.e.

$$\begin{cases} \partial_t u - \Delta u + \nabla \pi &= 0, \quad \text{in}\Omega \times (0,\infty), \\ \text{div} u &= 0, \quad \text{in } \Omega \times (0,\infty), \\ u(0) &= u_0, \quad \text{in } \Omega. \end{cases}$$

Here, u and π represent the velocity and pressure of the fluid, respectively.

We consider in particular domains $\Omega \subset \mathbb{R}^n$ for which the Helmholtz projection is known to exist only for certain values of p and sketch the proof of a recent result [4] which says that the Stokes operator admits maximal L^p-regularity on $L^p(\Omega)$ for a certain $p \in (1,\infty)$ provided the weak Neumann problem is solvable on these domains in the L^p-sense for this value of p. Furthermore, we discuss conditions implying that the Stokes operator admits a bounded H^∞-calculus on $L^p_\sigma(\Omega)$.

References

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L^p-Theory of the Navier-Stokes equations Kato iteration scheme, classical solutions

Matthias Hieber

TU Darmstadt

Department of Mathematics, Schlossgartenstr. 7, 64289 Darmstadt, Germany

In this talk we investigate the question of local and global solvability of the Navier-Stokes in domains $\Omega \subset \mathbb{R}^n$, i.e.

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla \pi &= 0, \quad \text{in}\,\Omega \times (0,\infty), \\ \text{div} u &= 0, \quad \text{in } \Omega \times (0,\infty), \\ u(0) &= u_0, \quad \text{in } \Omega. \end{cases}$$

This question has been considered by many authors in various scaling invariant spaces, in particular in

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow B_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow BMO^{-1}(\mathbb{R}^3) \hookrightarrow B_{\infty,\infty}^{-1}(\mathbb{R}^3),$$

where $3 < p < \infty$. The space $BMO^{-1}(\mathbb{R}^3)$ is the largest scaling invariant space known for which the above equation is well-posed. Topics we discuss include

- Gradient and $L^p - L^q$ smoothing properties of the Stokes semigroup,
- Kato iteration scheme,
- Mild and classical solutions
- local and global solutions for small data for $n = 2$ and $n = 3$.

References

[1] M. Geissert, M. Hess, M. Hieber, C. Schwarz, K. Stavrakidis), Maximal $L^p - L^q$-estimates for the Stokes equation: a short proof of Solonnikov's theorem. *J. Math. Fluid Mechanics*, **12** (2010), 47-60.

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Fluid-Solid Interactions: Part I: The case of Newtonian Fluids

Matthias Hieber

TU Darmstadt

Department of Mathematics, Schlossgartenstr. 7, 64289 Darmstadt, Germany

The study of the motions of rigid bodies immersed in a fluid is a classical problem of fluid mechanics. In this talk we develop an L^p-theory for strong solutions to the fluid-rigid body interaction problem, first for Newtonian fluids and later on in Part II for generalized Newtonian fluids. The L^p-theory developed in this first part is not only interesting for its own sake but it will be of central importance in the case of generalized Newtonian fluids where the assumption $p > 5$ will be needed.

The fluid's motion is governed by the equations,

$$\begin{cases} v_t + \operatorname{div} \mathbf{T}(v,q) + (v \cdot \nabla)v = f & \text{in } \mathcal{Q}_{\mathcal{D}}, \\ \operatorname{div} v = 0 & \text{in } \mathcal{Q}_{\mathcal{D}}, \\ v = v_{\mathcal{B}} & \text{on } \mathcal{Q}_{\Gamma}, \\ v(0) = v_0 & \text{in } \mathcal{D}(0), \end{cases}$$

where v and q denote the velocity and pressure of the fluid and $\mathbf{T}(v,q)$ its stress tensor. The fluid equations are coupled by the balance equations for the momentum and the angular momentum of the rigid body,

$$\begin{cases} \mathrm{m}\eta'(t) + \int_{\Gamma(t)} \mathbf{T}(v,q)(t,x)n(t,x)\,\mathrm{d}\sigma = \mathbf{F}(t), & t \in \mathbb{R}_+, \\ (J\omega)'(t) + \int_{\Gamma(t)} (x - x_c(t)) \times \mathbf{T}(v,q)(t,x)n(t,x)\,\mathrm{d}\sigma = \mathbf{M}(t), & t \in \mathbb{R}_+, \\ \eta(0) = \eta_0, & \\ \omega(0) = \omega_0, & \end{cases}$$

which contain the drag force and the torque exerted by the fluid onto the body. In this first part we show the existence of a unique, local solution in the case of a Newtonian fluid.

References

[1] K. Götze, M. Hieber, M. Geissert, L^p-theory of fluid-rigid body interactions for Newtonian and generalized Non-Newtonian fluids. *Trans. Amer. Math. Soc.*, appeared online

Fluid-Solid Interactions: Part II: The case of generalized Newtonian Fluids

Matthias Hieber

TU Darmstadt

Department of Mathematics, Schlossgartenstr. 7, 64289 Darmstadt, Germany

In this talk we continue our investigation of strong L^p-solutions to the fluid-rigid body interaction problem, now for generalized Newtonian fluids.

The fluid's motion is governed by the equations,

$$\begin{cases} v_t + \operatorname{div} \mathbf{T}(v,q) + (v\cdot\nabla)v = f & \text{in } \mathcal{Q}_{\mathcal{D}}, \\ \operatorname{div} v = 0 & \text{in } \mathcal{Q}_{\mathcal{D}}, \\ v = v_{\mathcal{B}} & \text{on } \mathcal{Q}_{\Gamma}, \\ v(0) = v_0 & \text{in } \mathcal{D}(0), \end{cases}$$

where the stress tensor $\mathbf{T}(v,q)$ is given by

$$\mathbf{T}^{\mu}(v,q) := \mu(|\mathcal{E}^{(v)}|_2^2)\mathcal{E}^{(v)} - q\mathrm{Id},$$

where for the viscosity μ we assume that $\mu \in C^{1,1}(\mathbb{R}_+;\mathbb{R})$ satisfying $\mu(s) > 0$ and $\mu(s) + 2s\mu'(s) > 0$ for all $s \geq 0$. The fluid equations are again coupled by the balance equations for the momentum and the angular momentum of the rigid body,

$$\begin{cases} \mathrm{m}\eta'(t) + \int_{\Gamma(t)} \mathbf{T}(v,q)(t,x)n(t,x)\,\mathrm{d}\sigma = \mathbf{F}(t), & t \in \mathbb{R}_+, \\ (J\omega)'(t) + \int_{\Gamma(t)} (x - x_c(t)) \times \mathbf{T}(v,q)(t,x)n(t,x)\,\mathrm{d}\sigma = \mathbf{M}(t), & t \in \mathbb{R}_+, \\ \eta(0) = \eta_0, \\ \omega(0) = \omega_0, \end{cases}$$

The dependence of the viscosity on the shear rate implies that the operator A given by

$$\begin{aligned} A(v)_i &:= (\operatorname{div}(\mu(|\mathcal{E}^{(v)}|_2^2)\mathcal{E}^{(v)}))_i \\ &= \mu(|\mathcal{E}^{(v)}|_2^2)\Delta v_i + 2\mu'(|\mathcal{E}^{(v)}|_2^2) \sum_{j,k,l=1}^{3} \varepsilon_{ij}^{(v)} \varepsilon_{kl}^{(v)} \partial_j \partial_l v_k, \end{aligned}$$

is quasi-linear. Freezing A at a reference solution v_* we obtain the linear operator A_* given by

$$(A_* v)_i = \mu(|\mathcal{E}^{(v_*)}|_2^2)\Delta v_i + 2\mu'(|\mathcal{E}^{(v_*)}|_2^2) \sum_{j,k,l=1}^{3} \varepsilon_{ij}^{(v_*)} \varepsilon_{kl}^{(v_*)} \partial_j \partial_l v_k$$

and investigate the above fluid rigid body interaction problem as a *quasilinear evolution equation.*

References

[1] K. Götze, M. Hieber, M. Geissert, L^p-theory of fluid-rigid body interactions for Newtonian and generalized Non-Newtonian fluids. *Trans. Amer. Math. Soc.*, appeared online

Geophysical Flows: Part I: The Equations of Navier-Stokes in the Rotational Framework

Matthias Hieber

TU Darmstadt

Department of Mathematics, Schlossgartenstr. 7, 64289 Darmstadt, Germany

Consider the equations of Navier-Stokes equations with Coriolis force, i.e.

$$\begin{aligned} u_t - \nu \Delta u + \omega e_3 \times u + u \cdot \nabla u + \nabla p &= f, \\ \text{div } u &= 0, \\ u(0) &= u_0, \end{aligned}$$

on all of $\mathbb{R}^3$, where ω denotes the speed of rotation and e_3 is the unit vector in the x_3-direction. If $\omega = 0$, the classical Navier-Stokes equations have been considered by many authors in various scaling invariant spaces, in particular in

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow B_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow BMO^{-1}(\mathbb{R}^3) \hookrightarrow B_{\infty,\infty}^{-1}(\mathbb{R}^3),$$

where $3 < p < \infty$. It is now a natural question to ask whether, for given and fixed $\omega > 0$, there exist global solutions to the above equation provided the initial data belong to some suitable function spaces. In this context, Hieber and Shibata proved a global well-posedness result for initial data being small with respect to $H^{\frac{1}{2}}(\mathbb{R}^3)$. Generalizations of this result to Fourier-Besov spaces are due to Konieczny and Yoneda, and Iwabuchi and Takada. In this talk, we continue this line of research and show that the above equations admit a unique, global solution provided the initial data are small in various Fourier-Besov spaces.

Moreover, we also investigate the two-dimensional setting and show that this case there exists a unique, global mild solution for all, (not necessarily small), $u_0 \in L^p(\mathbb{R}^2)$ for $2 \leq p < \infty$.

References

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Geophysical Flows: Part II: Stablity of Ekman Boundary Layers

Matthias Hieber

TU Darmstadt

Department of Mathematics, Schlossgartenstr. 7, 64289 Darmstadt, Germany

Consider the the initial value problem for the three dimensional Navier-Stokes equations with rotation in the half-space $\mathbb{R}^3_+$ subject to Dirichlet boundary conditions, i.e.

$$\begin{cases} \partial_t u - \nu\Delta u + \Omega \mathbf{e}_3 \times u + (u\cdot\nabla)u + \nabla p = 0, & t>0,\ x\in\mathbb{R}^3_+, \\ \text{div } u = 0, & t>0,\ x\in\mathbb{R}^3_+, \\ u(t,x_1,x_2,0) = 0, & t>0,\ x_1,x_2\in\mathbb{R}, \\ u(0,x) = u_0, & x\in\mathbb{R}^3_+, \end{cases}$$

Here, $\mathbf{e}_3$ denotes the unit vector in x_3-direction and the the constant $\Omega\in\mathbb{R}$ is called the Coriolis parameter. It is well known that the above system has a stationary solution which can be expressed even explicitly as

$$\begin{aligned} u_E(x_3) &= u_\infty(1-e^{-x_3/\delta}\cos(x_3/\delta), e^{-x_3/\delta}\sin(x_3/\delta), 0)^T, \\ p_E(x_2) &= -\Omega u_\infty x_2, \end{aligned}$$

where δ is defined by $\delta := (\frac{2\nu}{\Omega})^{1/2}$ and $u_\infty \geq 0$ is a constant. This stationary solution is called the *Ekman spiral.* We consider perturbations of the Ekman spiral by functions u solving the above equation. To this end, set

$$w := u - u_E, \quad \text{and} \quad q := p - p_E.$$

Since (u_E, p_E) is a stationary solution of (), the pair (w,q) formally satisfies the equations

$$\begin{cases} \partial_t w - \nu\Delta w + \Omega\mathbf{e}_3\times w + (u_E\cdot\nabla)w + w_3\partial_3 u_E + (w\cdot\nabla)w + \nabla q &= 0, & t>0,\ x\in\mathbb{R}^3_+, \\ \text{div } w &= 0, & t>0,\ x\in\mathbb{R}^3_+, \\ w(x_1,x_2,0) &= 0, & t>0,\ x_1,x_2\in\mathbb{R}, \\ w(0,x) &= w_0, & x\in\mathbb{R}^3_+, \end{cases}$$

where $w_0 = u_0 - u_E$.

In this talk we prove that the Ekman spiral is nonlinearly stable with respect to L^2-perturbations provided the corresponding Reynolds number is small enough.

References

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The Stokes Equation on Spaces of Bounded Functions

Matthias Hieber

TU Darmstadt

Department of Mathematics, Schlossgartenstr. 7, 64289 Darmstadt, Germany

In this talk we consider the resolvent approach to the Stokes operator in $L^\infty_\sigma(\Omega)$ for a large class of domains and prove a priori estimates for the resolvent problem

$$\begin{cases} \lambda v - \Delta v + \nabla q &= f \quad \text{in} \quad \Omega, \\ \operatorname{div} v &= 0 \quad \text{in} \quad \Omega, \\ v &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

of the form

$$\begin{aligned} |\lambda| ||v||_{L^\infty(\Omega)} + |\lambda|^{1/2} ||\nabla v||_{L^\infty(\Omega)} &+ |\lambda|^{n/2p} \sup_{x\in\Omega} ||\nabla^2 v||_{L^p(\Omega_{x,|\lambda|^{-1/2}})} \\ &+ |\lambda|^{n/2p} \sup_{x\in\Omega} ||\nabla q||_{L^p(\Omega_{x,|\lambda|^{-1/2}})} \leq C||f||_\infty. \end{aligned}$$

Here $f \in L^\infty_\sigma(\Omega)$, where $L^\infty_\sigma(\Omega)$ is defined for any open set $\Omega \subset \mathbb{R}^n$ as

$$L^\infty_\sigma(\Omega) := \{f \in L^\infty(\Omega) : \int_\Omega f \cdot \nabla \varphi dx = 0 \quad \text{for all} \quad \varphi \in \widehat{W}^{1,1}(\Omega)\},$$

where $\widehat{W}^{1,1}(\Omega) = \{\varphi \in L^1_{\text{loc}}(\Omega) : \nabla\varphi \in L^1(\Omega)\}$. Our approach is inspired by the Masuda-Stewart approach for elliptic operators.

Combining the above a priori estimate with a recent approximation argument due to Abe and Giga for bounded and exterior domains Ω, we obtain in particular that the Stokes operator A generates an analytic semigroup T on $L^\infty_\sigma(\Omega)$ of angle $\pi/2$ provided Ω is a bounded or exterior domain having C^3-boundary.

References

[1] K. Abe, Y. Giga, M. Hieber Stokes resolvent estimates for $L^\infty(\Omega)$. Preprint 2012.

[2] W. Desch, M. Hieber, J. Prüss, L^p-Theory of the Stokes operator in a half space *J. Evolution Eq.*, **1**, (2001), 115-142.