Banach space valued martingales and geometric properties of the space

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Convergence of L_1 bounded martingales in a Banach space

We will present different equivalent conditions on a Banach space F to the property that each L_1 bounded martingale with values in F is a.s. convergent. In the lectures we will restrict ourselves to martingales (f_n) in a Banach space F which are Bochner integrable.

Lemma 1 Let (f_n) be L_1 bounded martingale in a Banach space F. The following conditions are equivalent

1) (f_n) is a.s. convergent

2) (f_n) is convergent in laws

3) given a total family $D \subset F'$ a.s. the sequence $(f_n(\omega))$ contains a subsequence which is convergent in w_D topology on F

Definition 1 We say that F has *Radon-Nikodým property* (denoted RNP) if each σ-additive measure ν with values in F with finite variation has a density with respect to the variation.

Theorem 1 (Chatterji) Each L_1 bounded martingale in F is a.s. convergent if and only if F has RNP.

Corollary 1 If F is a reflexive, or more generally each separable subspace of F is ismorphic to a dual Banach space, then F has RNP.

Definition 2 We say that $A \subset F$ is dentable if for each $\epsilon > 0$ there is a hyperlane in F which cuts off A a nonempty subset with diameter less than ϵ .

Theorem 2 (Rieffel, Huff) Banach space F has RNP if and only if each bounded subset in F is dentable.

Definition 3 Let μ be a probability measure on a locally convex linear space F (defined on Borel subsets of F). We say that that a $m \in F$ is *barycenter* of μ if each $x' \in F'$ is integrable on F and $x'(m) = \int_F x'(u)d\mu(u)$. Such m if exists is unique.
Bounded, convex and closed $A \subset F$ is said to have *Choquet* proprerty if each $x \in A$ is barycenter of some probability measure supported by Borel subset contained in $Ext(A)$- the set of all extremal points of A.
Jf all such sets have Choquet property we say that F has Choquet property.

Theorem 3 (Choquet) Each compact, metrizable convex subset of locally convex linear space has Choquet property

Theorem 4 (Edgar) Each separable Banach space F with RNP has Choquet property.

Uwaga 1 Theorems 1,2,4 can be easily localized in the following sense: evrywere in their formulations we replace Banach space F by convex closed baunded subset of F.

As consequences of the above Theorems it is not hard to prove the following

Proposition 1 Each of the following is equivalent to RNP for a Banch space F

1. Each function $f : [0,1] \to F$ with bounded variation is a.e. differentiable,

2. For each absolutely continuous function $f : [0,1] \to F$ there is an integrable function $g : [0,1] \to F$ such that $f(t) = \int_0^t g(s)ds + f(0)$ for $t \in [0,1]$ (then $g(t) = f'(t)$ a.e.)

3. Each bounded operator $T : L_1(\Omega, \mathcal{F}, P) \to F$ is of the form $T(f) = \int_\Omega fgdP$ where $g : \Omega \to F$ is a $\mathcal{F}$- measurable, bounded function.

4. Each operator as above can be factorized through l_1, i.e. there are bounded operators $S : L_1(\Omega, \mathcal{F}, P) \to l_1$ and $R : l_1 \to F$ such that $T = RS$

5. The dual space to $L_p(F)$ is cannonically isomorphic to $L_q(F')$, for $\frac{1}{p} + \frac{1}{q} = 1, p < \infty$

The next Proposition is more difficult medskip

Proposition 2 If A is a bounded closed and convex subset of a Banach space F then each of the following is equivalent to RNP for A

1. Each subset of A is dentable.

2. Each closed and convex subset B of A contains points of dentability for B , i.e a point $x \in B$ such that for each $\epsilon > 0$, x is not the closure of $conv(B \setminus \{y \in F : ||y - x|| < \epsilon\})$.

3. Each subset B as above have strongly exposed point, i.e point x such that for some $x' \in F'$ and each $(x_n) \subset B$ the convergence $\lim_n x'(x_n) = x'(x)$ implies $\lim_n x_n = x$.

4. Each convex and closed subset of A is the closure of the convex hull of its strongly exposed points

Let us remind that KMP (Krein Milman Property) for a closed, convex bounded set $A \subset F$ means that B is the closure of $conv(Ext(B))$ for each closed convex $B \subset A$. Banach space F have KMP if each bounded convex and close subset of F has KMP.

For Banach spaces (as well for closed, convex and bounded sets) it is

$$\text{RNP} \Rightarrow \text{Choquet property} \Rightarrow \text{KMP}$$

It is longs tanding open problem if any of the inverse implications is true.

The following results are useful. The provided answers to open problems in the past.

Facts 1. There is a seprable Banach space with RNP which is not isomprphic to a subspace of a separable dual Banach space, i.e. the inverse to Corollary 1 is not true.

2. If F does not posses KMP then there is unifromly bounded martingale (f_n) in F and $\epsilon > 0$ such that $||f_n - f_{n-1}|| > \epsilon$ a.s.

3. There is a Banach space without RNP which does not contain Walsh-Palej martingale with the properties as above.

References

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