On regularity of weak solutions of the Navier-Stokes equations I

J. Neustupa, Prague

Abstract of the lecture

We assume that Ω is a domain in $\mathbb{R}^3$ and $T > 0$. We denote $Q_T := \Omega \times (0, T)$. We deal with the Navier–Stokes initial–boundary value problem

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} \qquad \text{in } Q_T, \tag{1}$$

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } Q_T, \tag{2}$$

$$\mathbf{v} = \mathbf{0} \qquad \text{on } \partial\Omega \times (0, T), \tag{3}$$

$$\mathbf{v} = \mathbf{v}_0 \qquad \text{in } \Omega \times \{0\}. \tag{4}$$

- The notion of the *Leray–Hopf weak solution* of problem (1)–(4) and the basic information on its existence and related properties (the L^2–weak continuity, an associated pressure, energy inequality and equality, strong energy inequality).
- Question of uniqueness of weak solutions of problem (1)–(4), known theorems, importance of the energy inequality in studies of uniqueness.
- Question of regularity of a weak solution of the problem (1)–(4) – one the the so called *millennium problems*.
- Leray's proposal for the construction of a singularity and the related negative result of Nečas, Růžička, Šverák [4].

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On regularity of weak solutions of the Navier-Stokes equations II

J. Neustupa, Prague

Abstract of the lecture

- Serrin's criterion for the interior spatial regularity of the weak solution $\mathbf{v}$, some generalizations. Recall that the criterion assumes that $\mathbf{v} \in L^r(t_1, t_2; \mathbf{L}^s(\Omega'))$, where $0 \leq t_1 < t_2 \leq T$ and $\Omega' \subset \Omega$, for certain exponents r and s satisfying the condition $2/r + 3/s \leq 1$.

- What one can say on regularity of the time derivative of $\mathbf{v}$ and the pressure in $\Omega' \times (t_1, t_2)$ under Serrin's conditions? Relation to the used boundary conditions.

- A Serrin–type criterion for the regularity of weak solution $\mathbf{v}$ in the whole domain Ω.

- A remark on the two–dimensional case: here, the weak solution automatically belongs to Serrin's regularity class.

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On regularity of weak solutions of the Navier-Stokes equations III

J. Neustupa, Prague

Abstract of the lecture

- Leray's "Theorème de Structure" on the partial regularity of a weak solution $\mathbf{v}$, satisfying the strong energy inequality. (The interval $(0, T)$ can be split to the union of a system of open intervals where the solution is "smooth" and a set Γ, whose 1–dimensional Lebesgue measure, or even $\frac{1}{2}$–dimensional Hausdorff measure, is zero.)
- Several definitions of the notion *regular point* of weak solution $\mathbf{v}$ (in the sense of [1], [2], [3], [4] and others), relations between various definitions.
- The notion of the so called *suitable weak solution* (in the sense of Caffarelli–Kohn–Nirenberg [1]).
- Generalized energy inequality.
- The Hausdorff dimension of the set of singular points of a suitable weak solution. Importance in the localization procedures.

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On regularity of weak solutions of the Navier-Stokes equations IV

J. Neustupa, Prague

Abstract of the lecture

- Several criteria for the local regularity of weak solution $\mathbf{v}$ (i.e. the criteria which guarantee that a chosen space–time point $(\mathbf{x}_0, t_0)$ is a regular point of solution $\mathbf{v}$). Our list involves the criteria from [1], [4], [6], [7] and [8]. Basic ideas of proofs of some of the criteria.
- Criteria for the local regularity of weak solution $\mathbf{v}$ that impose conditions on the pressure, respectively only on the negative part of pressure, see [5] and [3].
- Regularity criteria that impose conditions only on some components of the velocity.
- Regularity criteria that impose conditions only on some components of the vorticity or the gradient of velocity.

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Introduction to modelling of flows around rotating bodies I

J. Neustupa, Prague

Abstract of the lecture

Suppose that K is a compact body in $\mathbb{R}^3$, rotating about the x_1–axis with the angular velocity ω. Put $\boldsymbol{\omega} = \omega \mathbf{e}_1$ where $\mathbf{e}_1$ is the unit vector oriented in the direction of the x_1–axis. Denote further by $\Omega(t)$ the exterior of K at time t. Put

$$O(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega t & \sin\omega t \\ 0 & -\sin\omega t & \cos\omega t \end{pmatrix}$$

Then $\mathbf{x} \equiv (x_1, x_2, x_3) \in \Omega(t) \iff \mathbf{x}' \equiv O(t)\mathbf{x} \in \Omega(0)$. Thus, $\mathbf{x}'$ denotes the Cartesian coordinates connected with the rotating body. In order to get a problem in a fixed domain instead of the time–dependent domain $\Omega(t)$, many authors use the transformation

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) &= O^T(t)\,\mathbf{u}'(\mathbf{x}',t) = O^T(t)\,\mathbf{u}'\big(O(t)\mathbf{x},t\big), \\ p(\mathbf{x},t) &= p'(\mathbf{x}',t) = p'\big(O(t)\mathbf{x},t\big). \end{aligned}$$

Provided $\mathbf{u}$ satisfies the Navier–Stokes system in domain $\Omega(t)$, and p is the associated pressure, functions $\mathbf{u}'$, p' satisfy the system of equations

$$\partial_t \mathbf{u}' - \nu\Delta'\mathbf{u}' - (\boldsymbol{\omega}\times\mathbf{x}')\cdot\nabla'\mathbf{u}' + \boldsymbol{\omega}\times\mathbf{u}' + (\mathbf{u}'\cdot\nabla')\mathbf{u}' + \nabla' p' = \mathbf{f}' \qquad (1)$$

$$\nabla'\cdot\mathbf{u}' = 0 \qquad (2)$$

in the fixed domain $\Omega(0)$, where ∇', respectively Δ', denote the operator nabla, respectively the Laplace operator, with respect to $\mathbf{x}'$. If function $\mathbf{u}$ satisfies the no–slip boundary condition on the surface of body K, i.e. $\mathbf{u}(\mathbf{x},t) = \boldsymbol{\omega}\times\mathbf{x}$ (for $\mathbf{x}\in\partial\Omega(t)$) then function $\mathbf{u}'$ satisfies the condition

$$\mathbf{u}'(\mathbf{x}',t') = \boldsymbol{\omega}\times\mathbf{x}' \qquad \text{for } \mathbf{x}'\in\partial\Omega(0). \qquad (3)$$

We present fundamental qualitative properties of the problem (1), (2), (3), beginning with the linearized system and continuing with the nonlinear system.

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Introduction to modelling of flows around rotating bodies II

(Spectral analysis of associated linearized operators)

J. Neustupa, Prague

Abstract of the lecture

We come from the nonlinear system (1), (2) in domain $\Omega(0)$. In order to have a simple notation, we further omit the primes and we write only Ω instead of $\Omega(0)$.

By analogy with the classical Stokes operator, which plays a fundamental role in the analysis of the Navier–Stokes equations, now we have to deal with the Stokes–type operator

$$A^{\omega}\mathbf{u} \;:=\; \Pi_\sigma \nu \Delta \mathbf{u} + \Pi_\sigma [(\boldsymbol{\omega} \times \mathbf{x}) \cdot \nabla \mathbf{u} - \boldsymbol{\omega} \times \mathbf{u}], \tag{4}$$

respectively with the Oseen–type operator

$$L^{\omega}_{\gamma}\mathbf{u} \;:=\; A^{\omega}\mathbf{u} + \gamma\, \partial_1 \mathbf{u} \tag{5}$$

in the function space $L^2_\sigma(\Omega)$ (the subspace of $\mathbf{L}^2(\Omega)$, containing the so called solenoidal vector functions in Ω). Here, Π_σ denotes the orthogonal projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{L}^2_\sigma(\Omega)$.

We explain the notions of the nullity, deficiency, approximate nullity, approximate deficiency of a linear operator, Fredholm and semi–Fredholm operator, point spectrum, continuous spectrum, residual spectrum and the essential spectrum of a linear operator, and give the characterization of the spectrum of both the operators A^{ω} and L^{ω}_{γ}.

It is remarkable that, in contrast to the spectrum of the classical Stokes operator A^0 (which is the half–line covering the non-positive part of the real axis in the complex plane), the spectrum of the Stokes–type operator A^{ω} (for $\omega \neq 0$) consists of infinitely many half–lines parallel to the real axis. Similarly, while the spectrum of the classical Oseen operator L^0_γ covers a parabolic region in the complex plane, the spectrum of operator L^{ω}_{γ} (with $\omega \neq 0$) is a union of infinitely many such regions.

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