The Flux Problem in the Theory of Stationary Navier–Stokes Equations

Konstantin Pileckas
Faculty of Mathematics and Informatics,
Vilnius University

Lecture 1

We study the stationary Navier-Stokes system with homogeneous boundary conditions

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \\ \mathbf{v} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

It will be shown that the weak formulation of problem (1) is equivalent to the operator equation with a compact operator in a Hilbert space. Using the Leray–Schauder fixed point theorem the solvability of this operator equation will be proved.

Lecture 2

The stationary Navier-Stokes system with nonhomogeneous boundary conditions

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \\ \mathbf{v} = \mathbf{a} \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

will be studied in the multi-connected domain $\Omega = \Omega_0 \setminus \cup_{j=1}^{N}\Omega_j$, where $\bar{\Omega}_j \subset \Omega$, $\Omega_j \cap \Omega_i = \emptyset$, $j \neq i$.

The continuity equation $\operatorname{div}\mathbf{v} = 0$ implies the necessary compatibility condition for the solvability of problem (2):

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=1}^{N} \int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=1}^{N} F_j = 0, \tag{3}$$

where $\mathbf{n}$ is a unit vector of the outward (with respect to Ω) normal to $\partial\Omega$, $\Gamma_j = \partial\Omega_j$. The compatibility condition (3) means that the net flux of the fluid over the boundary $\partial\Omega$ is zero.

In this lecturer we shall prove the existence of the solution under the stronger than (3) condition which requires the all fluxes F_j of the boundary value $\mathbf{a}$ to be zero separately across each component Γ_j of the boundary $\partial\Omega$:

$$F_j = \int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = 0, \qquad j = 1, 2, \dots, N, \tag{4}$$

Notice that the condition (4) does not allow the presence of sinks and sources.

We describe the method based on of the Leray–Hopf's extension of the boundary data.

Lecture 3

We show that the Leray–Hopf's extension of the boundary data is not possible, when the condition (4) is violated (the counterexample of Takashita will be presented). We will describe a different method of getting an a priory estimate by a contradiction (under the condition (4)).

A counterexample of Ch. Amick will be presented.

Lecture 4

In this lecture we consider the problem (2) in a two dimensional symmetric domain. We suppose that the boundary value $\mathbf{a}$ is also symmetric, but we we do not assume the fluxes F_j to be zero. For the symmetric case the existence of the solution will be proved only under the necessary condition (3).

In this lecture we also prove the existence of the solution to problem (2) in the case of "sufficiently small" fluxes F_j.

Lecture 5

We study the nonhomogeneous boundary value problem for Navier–Stokes equations in a two–dimensional bounded multiply connected domain $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, $\overline{\Omega}_2 \subset \Omega_1$. It will proved that this problem has a solution if the flux F of the boundary value through $\partial\Omega_2$ is nonnegative. Notice that we do not impose any smallness assumptions on the value of $|F|$. The proof of the main result uses the Bernoulli law for a weak solution to the Euler equations and the one-side maximum principle for the total head pressure (Bernoulli function) corresponding to this solution.

Lecture 6

In this lecture the detailed proof of the Bernoulli law for a weak solution to the Euler equations will be presented.

Lecture 7

In this lecture the detailed proof of the one-side maximum principle for the total head pressure (Bernoulli function) corresponding to a weak solution of the Euler equations will be presented.

Lecture 8

The nonhomogeneous boundary value problem (2) for the steady Navier–Stokes equations will be studied in a two–dimensional exterior domain Ω. It is assumed that the domain Ω and the boundary value $\mathbf{a}$ are symmetric with respect to the x_1-axis. The existence of a solution to this problem will be proved for arbitrary values of the flux F of the boundary datum.

Lecture 9

In this lecture we discuss the generalizations of obtained in the two-dimensional case results to the three-dimensional bounded and exterior axially symmetric domains.

Lecture 10

In the last lecture we discuss stationary Navier–Stokes system with nonhomogeneous boundary conditions in a class of domains Ω with noncompact boundary $\partial\Omega$ which is multiply connected and consists of M infinite connected components S_m, which form the outer boundary, and I compact connected components Γ_i forming the inner boundary Γ. The boundary value $\mathbf{a}$ is assumed to have a compact support and it is supposed that the fluxes of $\mathbf{a}$ over the components Γ_i of the inner boundary are sufficiently small. However, no restrictions are imposed on fluxes of $\mathbf{a}$ over the infinite components S_m. The existence of at least one weak solution to the Navier–Stokes problem will be proved. The obtained solution may have finite or infinite Dirichlet integral depending on geometrical properties of outlets to infinity.