On the discrete cube

Krzysztof Oleszkiewicz

Institute of Mathematics, University of Warsaw

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Discrete cube

The discrete cube $\{-1,1\}^n$ is one of the basic structures in:

- functional analysis and convex geometry:
  as a set of extreme points (vertices) of the unit cube $[-1,1]^n$
- probability theory: as a useful way to describe independent ± 1 symmetric random variables
- combinatorics:
  as a useful way to describe subsets of $\{1,2,\ldots,n\}$
- harmonic analysis: as a compact abelian group
  (with coordinate-wise multiplication)
- theoretical computer science:
  as a model for a collection of n bits (input data)

L^2 structure

We will denote $\{1, 2, \dots, n\}$ by $[n]$.

We equip $\{-1, 1\}^n$ with the uniform product probability measure μ.

For $f : \{-1, 1\}^n \longrightarrow \mathbf{R}$ we define its expectation:

$$\mathbf{E}[f] = \int_{\{-1,1\}^n} f \, d\mu = 2^{-n} \sum_{x \in \{-1,1\}^n} f(x).$$

We define the scalar product of functions $f, g : \{-1, 1\}^n \longrightarrow \mathbf{R}$ by

$$\langle f, g \rangle = \mathbf{E}[f \cdot g] = 2^{-n} \cdot \sum_{x \in \{-1,1\}^n} f(x) g(x).$$

$\mathcal{H} = L^2(\{-1, 1\}^n, \mu)$ is a 2^n-dimensional Hilbert space.

Walsh-Fourier system

For $A \subseteq [n]$ let $w_A : \{-1,1\}^n \longrightarrow \mathbf{R}$ be given by

$$w_A(x) = \prod_{i \in A} x_i,$$

with $w_\emptyset \equiv 1$. The Walsh functions $(w_A)_{A \subseteq [n]}$ form an orthonormal basis of $\mathcal{H}$ which is called the Walsh-Fourier system.

Indeed, $\mathbf{E}[w_A] = 0$ unless $A = \emptyset$, and $w_A \cdot w_B = w_{A \Delta B}$, so that $\langle w_A, w_A \rangle = \mathbf{E}[w_\emptyset] = 1$, and $\langle A, B \rangle = \mathbf{E}[w_A \cdot w_B] = 0$ for $A \neq B$.

This proves the orthogonality of the Walsh-Fourier system.
Its completeness follows from the fact that the number of the Walsh functions on $\{-1,1\}^n$ is equal to $\dim(\mathcal{H}) = 2^n$.

Heat semigroup

For $t \geq 0$ let $T_t : \mathcal{H} \longrightarrow \mathcal{H}$ be a linear operator defined by

$$T_t w_A = e^{-|A|t} w_A.$$

The operators $(T_t)_{t \geq 0}$ form a semigroup:

$$T_{t+s} = T_t \circ T_s$$

for $t, s \geq 0$.

Obviously, $(T_t)_{t \geq 0}$ are contractions:

$$\|T_t f\|_{\mathcal{H}}^2 = \sum_{A \subseteq [n]} e^{-2|A|t} a_A^2 \leq \sum_{A \subseteq [n]} a_A^2 = \|f\|_{\mathcal{H}}^2,$$

where $(a_A)_{A \subseteq [n]}$ are the Walsh-Fourier coefficients of f:

$$a_A = \langle f, w_A \rangle,$$

so that $f = \sum_{A \subseteq [n]} a_A w_A$.

Generator

We define $L : \mathcal{H} \longrightarrow \mathcal{H}$ by

$$Lf = -\frac{d}{dt}\, T_t f\Big|_{t=0},$$

so that

$$Lw_A = |A| w_A$$

L is linear, and we can write $T_t = e^{-tL}$.

Obviously, L is positive semidefinite:

$$\langle f, Lf \rangle = \sum_{A\subseteq[n]} |A| \cdot a_A^2 \geq 0$$

for $f = \sum_{A\subseteq[n]} a_A w_A$.

Markov process and hypercontractivity

We will prove that the heat semigroup $(T_t)_{t\geq 0}$ is related to a continuous time $\{-1,1\}^n$-valued Markov process which is time and space homogenous. We will investigate its relation to the standard dicrete time symmetric random walk on the discrete cube. Then we will see that the heat semigroup in not only contractive in L^2 but also in L^p for every $p \geq 1$, and even more: it is hypercontractive, i.e. for every $p > q > 1$ and $t \geq (\ln(p-1) - \ln(q-1))/2$ for every $f \in \mathcal{H}$ the celebrated Bonami-Beckner inequality

$$\|T_t f\|_p \leq \|f\|_q$$

holds true. We will prove this result and discuss its applications.