# The "Flux Leray's Problem" in the Theory of Navier–Stokes Equations

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# Lecture 1

During this lecture we study in a bounded domain $\Omega \subset \mathbb{R}^n$ the stationary Navier-Stokes system with homogeneous boundary conditions

$$\begin{cases} -\nu\Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \\ \mathbf{v} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_n}\right)$, $\mathbf{a}\cdot\mathbf{b} = \sum\limits_{i=1}^{n} a_i b_i$, $\operatorname{div} \mathbf{v} = \nabla\cdot\mathbf{v}$, $\Delta = \nabla\cdot\nabla$ is the Laplacian, $\mathbf{v}$ and $p$ stand for the velocity vector and for the pressure, $\mathbf{f}$ is the density of external forces:

$$\mathbf{v} = (v_1, \dots, v_n), \qquad \mathbf{f} = (f_1, \dots, f_n),$$

$\nu > 0$ means the constant viscosity of the liquid.

Let $H(\Omega)$ be a subspace of solenoidal vector fields belonging to $\overset{\circ}{W}{}^1_2(\Omega)$. By a weak solution of problem (1) we understand a vector function $\mathbf{v} \in H(\Omega)$ satisfying the integral identity

$$\nu \int\limits_{\Omega} \nabla\mathbf{v}\cdot\nabla\boldsymbol{\eta}\, dx + \int\limits_{\Omega} (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\eta}\, dx = \int\limits_{\Omega} \mathbf{f}\cdot\boldsymbol{\eta}\, dx \quad \forall\, \boldsymbol{\eta} \in H(\Omega), \tag{2}$$

where

$$\nabla\mathbf{v}\cdot\nabla\boldsymbol{\eta} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial v_i}{\partial x_j}\frac{\partial \eta_i}{\partial x_j}.$$

It will be shown that the integral identity (2) is equivalent to the operator equation in the space $H(\Omega)$

$$\mathbf{v} = \mathcal{A}\mathbf{v} \tag{3}$$

with the compact operator $\mathcal{A}$. In order to apply the Leray–Schauder Fixed Point Theorem, we will prove that all possible solutions $\mathbf{v}^\lambda$ of the equation

$$\mathbf{v}^\lambda = \lambda\mathcal{A}\mathbf{v}^\lambda, \quad \lambda \in [0,\, 1], \tag{4}$$

are uniformly (with respect to $\lambda$) bounded. Then from the Leray–Schauder it follows that equation (3) has at least one solution.

The solution of the integral identity (2) (or, equivalently, of the operator equation (3)) in general could be non-unique. However we prove the uniqueness of this solution for small data.

We also show that there exists a pressure function $p \in L_2(\Omega)$ such that $\int\limits_\Omega p(x)\, dx = 0$ and

$$\nu \int\limits_\Omega \nabla \mathbf{v} \cdot \nabla \boldsymbol{\eta}\, dx + \int\limits_\Omega (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \boldsymbol{\eta}\, dx = \int\limits_\Omega p \,\mathrm{div}\, \boldsymbol{\eta}\, dx$$

$$+ \int\limits_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\, dx \quad \forall\, \boldsymbol{\eta} \in \overset{\circ}{W}{}^1_2(\Omega). \tag{5}$$

Moreover,

$$\mathbf{v} \in W^2_{2,loc}(\Omega), \quad p \in W^1_{2,loc}(\Omega)$$

and the pair $(\mathbf{v}, p)$ satisfies the Navier–Stokes equations (1) almost everywhere in $\Omega$.

In order to find the pressure function $p$ we would need to study the following auxiliary problem ("divergence problem"):

*For a given $g \in L_2(\Omega)$ with $\int\limits_\Omega g(x)\, dx = 0$ to find a function $\mathbf{w} \in \overset{\circ}{W}{}^1_2(\Omega)$ satisfying the equation*

$$\mathrm{div}\, \mathbf{w} = g \tag{6}$$

*and the inequality*

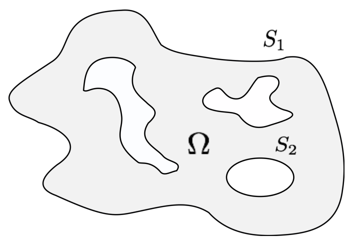
$$\|\nabla \mathbf{w}\|_{L_2(\Omega)} \le c\|g\|_{L_2(\Omega)}. \tag{7}$$

# Lecture 2

The stationary Navier-Stokes system with nonhomogeneous boundary conditions

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{a} & \text{on } \partial\Omega, \end{cases} \tag{8}$$

will be studied in a domain $\Omega = \Omega_0 \setminus \bigcup\limits_{j=1}^{N} \Omega_j$ with the multiply connected boundary. Here $\bar{\Omega}_j \subset \Omega$, $\Omega_j \cap \Omega_i = \emptyset$, $j \neq i$.




The continuity equation $\operatorname{div}\mathbf{v} = 0$ implies the necessary compatibility condition for the solvability of problem (8):

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\, dS = \sum_{j=1}^{N} \int\limits_{S_j} \mathbf{a}\cdot\mathbf{n}\, dS = \sum_{j=1}^{N} F_j = 0, \tag{9}$$

where $\mathbf{n}$ is a unit vector of the outward (with respect to $\Omega$) normal to $\partial\Omega$, $S_j = \partial\Omega_j$. The compatibility condition (9) means that the net flux of the fluid over the boundary $\partial\Omega$ is zero.

In this lecturer we shall prove the existence of the solution under the stronger than (9) condition which requires the all fluxes $F_j$ of the boundary value $\mathbf{a}$ to be zero separately across each component $S_j$ of the boundary $\partial\Omega$:

$$F_j = \int\limits_{S_j} \mathbf{a}\cdot\mathbf{n}\, dS = 0, \qquad j = 1, 2, \ldots, N, \tag{10}$$

Notice that the condition (10) does not allow the presence of sinks and sources.

First we will study the method based on of the Leray–Hopf's extension of the boundary value $\mathbf{a}$. If the condition (10) is valid, then there exists a function $\mathbf{b}$ such that

$$\operatorname{rot} \mathbf{b}(x)\big|_{\partial\Omega} = \mathbf{a}(x).$$

We construct the Leray–Hopf's cut–off function $\zeta(x, \varepsilon)$ having the following properties:

(i) $\zeta(x, \varepsilon) = 1 \ \ for \ \ x \in \partial\Omega, \quad \zeta(x, \varepsilon) = 0 \ \ for \ \ dist(x, \partial\Omega) \geq \delta = \delta(\varepsilon)$,

(ii) $0 \leq \zeta(x, \varepsilon) \leq 1$,

(iii) $|\nabla\zeta(x, \varepsilon)| \leq \dfrac{c\varepsilon}{dist(x, \partial\Omega)}$ with the constant $c$ independent of $\varepsilon$.

The Leray–Hopf's extension function has the form

$$\mathbf{B}(x, \varepsilon) = \operatorname{rot}\big(\zeta(x, \varepsilon)\mathbf{b}(x)\big). \tag{11}$$

Then

$$\operatorname{div} \mathbf{B}(x, \varepsilon) = 0, \quad \mathbf{B}(x, \varepsilon)\big|_{\partial\Omega} = \mathbf{a}(x).$$

We look for a week solution of the problem (8) in the form $\mathbf{v} = \mathbf{u} + \mathbf{B}$, where $\mathbf{u} \in H(\Omega)$. Then for $\mathbf{u}$ we get the integral identity

$$\nu \int\limits_\Omega \nabla\mathbf{u} \cdot \nabla\boldsymbol{\eta}\, dx + \int\limits_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \boldsymbol{\eta}\, dx + \int\limits_\Omega (\mathbf{u} \cdot \nabla)\mathbf{B} \cdot \boldsymbol{\eta}\, dx + \int\limits_\Omega (\mathbf{B} \cdot \nabla)\mathbf{u} \cdot \boldsymbol{\eta}\, dx$$

$$= \int\limits_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\, dx - \int\limits_\Omega (\mathbf{B} \cdot \nabla)\mathbf{B} \cdot \boldsymbol{\eta}\, dx - \nu \int\limits_\Omega \nabla\mathbf{B} \cdot \nabla\boldsymbol{\eta}\, dx \quad \forall\, \boldsymbol{\eta} \in H(\Omega). \tag{12}$$

The integral identity (12) is equivalent to the operator equation

$$\mathbf{u} = \mathcal{B}\mathbf{u}$$

with the compact operator $\mathcal{B}$ in the space $H(\Omega)$. In order to show that all possible solutions of the operator equation with the parameter $\lambda$ are uniformly bounded, we prove and apply the following Leray–Hopf's inequality

$$\Big| \int\limits_\Omega (\mathbf{u} \cdot \nabla)\mathbf{B} \cdot \mathbf{u}\, dx \Big| \leq c\varepsilon \int\limits_\Omega |\nabla\mathbf{u}|^2\, dx \quad \forall \mathbf{u} \in H(\Omega), \tag{13}$$

where the constant $c$ is independent of $\varepsilon$ and $\mathbf{u}$. We get the desired estimate of the solution by choosing in (13) the parameter $\varepsilon > 0$ sufficiently small.

# Lecture 3

We shall show that the Leray–Hopf's extension of the boundary data is not possible, if the condition (10) is violated (the counterexample of Takashita will be presented), i.e., we prove that if the fluxes $F_i$ of the boundary value $\mathbf{a}$ throw the connected components $S_i$ of the boundary $\partial\Omega$ are nonzero, then the Leray–Hopf's inequality (13), in general, is not valid. Thus in this case it is not possible to apply the same as in the previous lecture method.

In this lecture we consider the method of getting an a priory estimate by a contradiction. In order to simplify the proofs, we still assume for a while that the condition (10) is fulfilled. The main idea of the last method consist in the following. Consider the integral identity corresponding to the operator equation with the parameter $\lambda$:

$$\nu\int\limits_\Omega \nabla\mathbf{u}^\lambda\cdot\nabla\boldsymbol{\eta}\,dx-\lambda\int\limits_\Omega \big((\mathbf{u}^\lambda+\mathbf{B})\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{u}^\lambda\,dx-\lambda\int\limits_\Omega \big(\mathbf{u}^\lambda\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{B}\,dx$$

$$=\lambda\Big(\int\limits_\Omega \big(\mathbf{B}\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{B}\,dx-\nu\int\limits_\Omega \nabla\mathbf{B}\cdot\nabla\boldsymbol{\eta}\,dx+\nu\int\limits_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\,dx\Big)\qquad\forall\,\boldsymbol{\eta}\in H(\Omega).\quad(14)$$

Here $\mathbf{B}$ is an arbitrary divergence free extension of the boundary value $\mathbf{a}$.

Assume that the solutions of (14) are not uniformly bounded in $H(\Omega)$ with respect to $\lambda\in[0,1]$. Then there exist sequences $\{\lambda_k\}_{k\in\mathbb{N}}\subset[0,1]$ and $\{\mathbf{u}^{\lambda_k}=\mathbf{u}_k\}_{k\in\mathbb{N}}\in H(\Omega)$ such that

$$\nu\int\limits_\Omega \nabla\mathbf{u}_k\cdot\nabla\boldsymbol{\eta}\,dx-\lambda_k\int\limits_\Omega \big((\mathbf{u}_k+\mathbf{B})\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{u}_k\,dx-\lambda_k\int\limits_\Omega \big(\mathbf{u}_k\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{B}\,dx$$

$$=\lambda_k\Big(\int\limits_\Omega \big(\mathbf{B}\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{B}\,dx-\nu\int\limits_\Omega \nabla\mathbf{B}\cdot\nabla\boldsymbol{\eta}\,dx+\nu\int\limits_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\,dx\Big)\quad\forall\,\boldsymbol{\eta}\in H(\Omega),\quad(15)$$

and

$$\lim_{k\to\infty}\lambda_k=\lambda_0\in[0,1],\quad \lim_{k\to\infty}J_k=\lim_{k\to\infty}\|\nabla\mathbf{u}_k\|_{L_2(\Omega)}=\infty.$$

First, taking in (15) $\boldsymbol{\eta}=J_k^{-2}\mathbf{u}_k$ and passing to a limit we obtain the equality

$$\nu=\lambda_0\int\limits_\Omega \big(\widehat{\mathbf{u}}\cdot\nabla\big)\widehat{\mathbf{u}}\cdot\mathbf{B}\,dx,\qquad(16)$$

where $\widehat{\mathbf{u}}$ is a weak limit in $H(\Omega)$ of the sequence $\{J_k^{-1}\mathbf{u}_k\}$.

Second, taking in (15) $\boldsymbol{\eta} = J_k^{-2}\boldsymbol{\xi}$ with arbitrary $\boldsymbol{\xi} \in H(\Omega)$ we obtain that $\widehat{\mathbf{u}}$ and the corresponding pressure function $\widehat{p}$ satisfy the Euler equations:

$$\begin{cases} \lambda_0\big(\widehat{\mathbf{u}}\cdot\nabla\big)\widehat{\mathbf{u}} + \nabla\widehat{p} &= 0,\\ \text{div}\ \widehat{\mathbf{u}} &= 0,\\ \widehat{\mathbf{u}}|_{\partial\Omega} &= 0. \end{cases} \tag{17}$$

It follows from (17) that

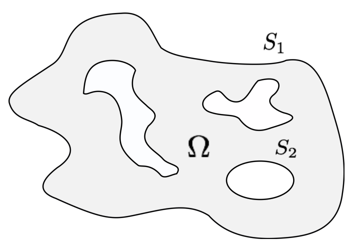
$$\widehat{p}\,|_{S_i} = \widehat{p}_i, i = 1, \dots, N,$$

where $\widehat{p}_i$ are constants.

Multiplying (17) by $\mathbf{B}$ and integrating by parts yields

$$\lambda_0 \int\limits_\Omega \big(\widehat{\mathbf{u}}\cdot\nabla\big)\widehat{\mathbf{u}}\cdot\mathbf{B}\,dx = \sum_{i=1}^N \widehat{p}_i F_i. \tag{18}$$

If the condition (10) is valid, i.e. $F_i = 0, i = 1, \dots, N$, then the right-hand side of (18) is equal to zero and (18) contradicts to (16). Thus, all possible solutions of (15) are bounded and by the Leray-Schauder Fixed Point Theorem there exists at least one weak solution of problem (8).




Since by incompressibility of the fluid

$$\sum_{i=1}^N F_i = 0,$$

the right-hand side of (18) is zero also in the case when

$$\widehat{p}_1 = \widehat{p}_2 = \ldots = \widehat{p}_N.$$

However, Ch. Amick has constructed a counterexample showing that, in general, this is not true.

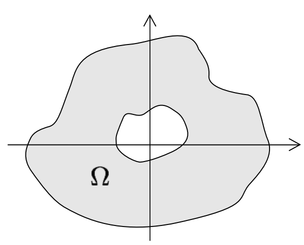
# Lecture 4

In this lecture we prove the existence of the solution to problem (8) in a bounded multiply connected domain $\Omega \in \mathbb{R}^n$ assuming that the fluxes $F_i, i = 1, \dots, N$, are "sufficiently small". More precisely, we prove that there exists a number $F_* = F_*(\nu) > 0$, dependent on the viscosity coefficient $\nu$, such that for

$$|F_i| \leq F_*, \;\; i = 1, \dots, N,$$

the problem (8) admits at least one weak solution. Note that here we do not assume the norms of the boundary value $\mathbf{a}$ and the external force $\mathbf{f}$ to be small. We prove this result using both proposed in the previous lectures approachers, i.e., by the special construction of the extension $\mathbf{B}$ of the boundary value $\mathbf{a}$, and by getting the a priory estimate by a contradiction.

Next, we consider problem (8) in a plane domain $\Omega$ with two components of the boundary $S_1$ and $S_2$. Assuming that $\mathbf{a} = F\nabla u_0 + \boldsymbol{\alpha}$, where $F \in \mathbb{R}$, $u_0$ is a harmonic function, and $\boldsymbol{\alpha}$ satisfies condition (10) (i.e., fluxes of $\boldsymbol{\alpha}$ over all $S_i$ are equal to zero), we prove that there is a countable subset $\mathcal{N} \subset \mathbb{R}$ such that if $F \notin \mathcal{N}$ and $\boldsymbol{\alpha}$ is small (in a suitable norm), then the problem (8) admits at least one weak solution. Moreover, if $\Omega \subset \mathbb{R}^2$ is an annulus and $u_0 = \log|x|$, then $\mathcal{N} = \emptyset$.



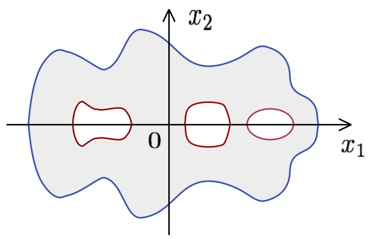

In the last part of the lecture we study the problem (8) in a two dimensional symmetric bounded domain, i.e., $\Omega$ is a bounded domain in $\mathbb{R}^2$ with multiply connected Lipschitz boundary $\partial\Omega$ consisting of $N$ disjoint components $S_j$: $\partial\Omega = S_1 \cup \ldots \cup S_N$ and $S_i \cap S_j = \emptyset$, $i \neq j$. Suppose that $\Omega$ is symmetric with respect to $x_1$–axis:

$$(x_1, x_2) \in \Omega \Leftrightarrow (x_1, -x_2) \in \Omega,$$

and assume that all components $S_j$ of the boundary $\partial\Omega$ intersect this axis.

We say that the vector function $\mathbf{u}(x_1, x_2)$ is symmetric, if $u_1$ is an even function of $x_2$ and $u_2$ is odd function of $x_2$ and $u_2$:

$$u_1(x_1, x_2) = u_1(x_1, -x_2), \quad u_2(x_1, x_2) = -u_2(x_1, -x_2).$$




Assuming that the boundary value $\mathbf{a}$ is symmetric we prove the existence of at least one weak solution of problem (8). Note that in the symmetric case we do not assume the fluxes $F_i$ to be zero or "small". The fluxes $F_i$ have to satisfy only the necessary compatibility condition (9). We prove this result by getting a priory estimate by a contradiction and we get this contradiction showing that in the symmetric case for the pressure $\widehat{p}(x)$ in Euler equations (17) holds the relations

$$\widehat{p}_1 = \widehat{p}_2 = \ldots = \widehat{p}_N = const,$$

where $\widehat{p}_i = \widehat{p}(x)|_{S_i}, i = 1, \ldots, N$.

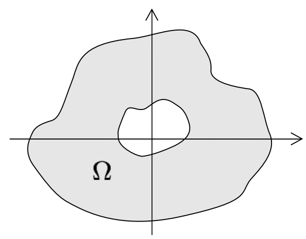
Finally, we show that in the symmetric case also the Leray–Hopf's extension of the boundary value $\mathbf{a}$ could be constructed.

# Lecture 5

In this lecture we study problem (8) in a plane domain

$$\Omega = \Omega_1 \setminus \overline{\Omega}_2, \quad \overline{\Omega}_2 \subset \Omega_1,$$

where $\Omega_1$ and $\Omega_2$ are bounded simply connected domains of $\mathbb{R}^2$ with Lipschitz boundaries $\partial\Omega_1 = S_1$, $\partial\Omega_2 = S_2$. Without loss of generality we may assume that $\Omega_2 \supset \{x \in \mathbb{R}^2 : |x| < 1\}$.




Since $\Omega$ has only two components of the boundary, condition (9) may be rewritten in the form

$$F = \int\limits_{S_2} \mathbf{a} \cdot \mathbf{n}\, dS = -\int\limits_{S_1} \mathbf{a} \cdot \mathbf{n}\, dS$$

($\mathbf{n}$ is an outward normal with respect to the domain $\Omega$). We prove the solvability of problem (8) without any restriction on the value of $|F|$ provided that $F > 0$. Since it is known (was proved in the previous lecture) that problem (8) is solvable for sufficiently small $|F|$, we conclude that the solution exists if $F \in [-F_*, \infty)$, where $F_*$ is some positive number. Note that this recent result is the first result on Leray's problem which does not require smallness of the net flux or symmetry conditions on the domain and boundary value. The proposed here method works only for $F > 0$. We do not have neither physical nor mathematical arguments for the existence or nonexistence of the solution to (8) in the case $F < 0$ with large $|F|$.

The proof of the existence theorem is based on a priory estimate which we obtain using the *reductio ad absurdum* argument which was described in previous lectures. The essentially new part in this argument is the use of the weak one-side maximum principle for the total head pressure (Bernoulli function) corresponding to weak solutions of the Euler equations and a representation of the total-head pressure in the divergence form. The proof of the above maximum principle is based on the Bernoulli Law for a weak solution to Euler equations.

Let $(\widehat{\mathbf{u}},\widehat{p}) \in W_2^1(\Omega)\times W_s^1(\Omega)$, $s\in[1,2)$, be a solution of the Euler system

$$\begin{cases} \big(\widehat{\mathbf{u}}\cdot\nabla\big)\widehat{\mathbf{u}}+\nabla\widehat{p} &= 0,\\ \operatorname{div}\widehat{\mathbf{u}} &= 0 \end{cases}$$

in a bounded domain $\Omega\subset\mathbb{R}^2$ with Lipschitz boundary ( $(\widehat{\mathbf{u}},\widehat{p})$ satisfies the Euler system for almost all $x\in\Omega$). Assume that

$$\int\limits_{S_i}\widehat{\mathbf{u}}\cdot\mathbf{n}dS=0,\ i=1,2,\dots,N.$$

Then there exists a stream function $\psi\in W_2^2(\Omega)$ such that $\nabla\psi=(-\widehat{u}_2,\widehat{u}_1)$ (note that by Sobolev Embedding Theorem $\psi$ is continuous in $\overline{\Omega}$) . Denote by $\Phi=\widehat{p}+\dfrac{|\widehat{\mathbf{u}}|^2}{2}$ the total head pressure corresponding to the solution $(\widehat{\mathbf{u}},\widehat{p})$. Obviously, $\Phi\in W_s^1(\Omega)$ for all $s\in[1,2)$. By direct calculations one easily gets the identity

$$\nabla\Phi\equiv\Big(\frac{\partial\widehat{u}_2}{\partial x_1}-\frac{\partial\widehat{u}_1}{\partial x_2}\Big)\big(\widehat{u}_2,-\widehat{u}_1\big)=(\Delta\psi)\nabla\psi. \tag{19}$$

If all functions are smooth, then from (19) the classical Bernoulli law follows immediately:

*The total head pressure $\Phi(x)$ is constant along any streamline of the flow.*

In the general case the following assertion holds.

**THEOREM 1.** *Let $\Omega\subset\mathbb{R}^2$ be a bounded multiply connected domain with Lipschitz boundary $\partial\Omega=\bigcup\limits_{i=1}^N S_i$. Assume that $\widehat{\mathbf{u}}\in W_2^1(\Omega)$ and $\widehat{p}\in W_s^1(\Omega)$, $s\in[1,2)$, satisfy Euler equations for almost all $x\in\Omega$ and*

$$\int\limits_{S_i}\widehat{\mathbf{u}}\cdot\mathbf{n}dS=0,\ i=1,\dots,N.$$

*Then for any connected set $K\subset\overline{\Omega}$ such that*

$$\psi\big|_K=\text{const},$$

*there exists a constant $C = C(K)$ such that*

$$\Phi(x) = C \quad \textit{for } \mathfrak{H}^1\textit{-almost all } x \in K.$$

*Here $\mathfrak{H}^1$ is the one dimensional Hausdorff measure.*

# Lecture 6

In this lecture the detailed proof of the Bernoulli law for a weak solution to the Euler equations will be presented, i.e., we prove the following theorem:

**THEOREM 2.** *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded multiply connected domain with Lipschitz boundary* $\partial\Omega = \bigcup\limits_{i=1}^{N} S_i$. *Assume that* $\widehat{\mathbf{u}} \in W_2^1(\Omega)$ *and* $\widehat{p} \in W_s^1(\Omega)$, $s \in [1,2)$, *satisfy Euler equations for almost all* $x \in \Omega$ *and*

$$\int\limits_{S_i} \widehat{\mathbf{u}} \cdot \mathbf{n} dS = 0,\ i = 1, \dots, N.$$

*Then for any connected set* $K \subset \overline{\Omega}$ *such that*

$$\psi\big|_K = \text{const},$$

*there exists a constant* $C = C(K)$ *such that*

$$\Phi(x) = C \quad \textit{for } \mathfrak{H}^1\textit{-almost all } x \in K.$$

*Here we denote by* $\mathfrak{H}^1$ *the one-dimensional Hausdorff measure, i.e.,*

$$\mathfrak{H}^1(F) = \lim_{t \to 0+} \mathfrak{H}^1_t(F),$$

*where* $\mathfrak{H}^1_t(F) = \inf\{\sum\limits_{i=1}^{\infty} \text{diam} F_i : \text{diam} F_i \le t, F \subset \bigcup\limits_{i=1}^{\infty} F_i\}$.

The proof of the theorem is based on properties of functions from the Sobolev space $W_1^2$ and on Morse-Sard and Luzin N-properties of Sobolev functions from $W_1^2$. For example, we shall use the following

**LEMMA.** (J. R. Dorronsoro) *Let* $\psi \in W_1^2(\mathbb{R}^2)$. *Then there exists a set* $A_\psi$ *such that* $\mathfrak{H}^1(A_\psi) = 0$, *and for all* $x \in \mathbb{R}^2 \setminus A_\psi$ *the function* $\psi$ *is differentiable (in the classical sense) at the point* $x$, *furthermore, the classical derivative coincides with* $\nabla\psi(x)$.

The results concerning Morse-Sard and Luzin N-properties were obtained recently by J. Bourgain, M. Korobkov and J. Kristensen:

**THEOREM 3.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $\psi \in W^2_1(\Omega)$. Then*

(i) *for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $U \subset \overline{\Omega}$ with $\mathfrak{H}^1_\infty(U) < \delta$ the inequality $\mathfrak{H}^1(\psi(U)) < \varepsilon$ holds;*

(ii) *for every $\varepsilon > 0$ there exists an open set $V \subset \mathbb{R}$ and a function $g \in C^1(\mathbb{R}^2)$ such that $\mathfrak{H}^1(V) < \varepsilon$, and for each $x \in \overline{\Omega}$ if $\psi(x) \notin V$ then $x \notin A_\psi$, the function $\psi$ is differentiable at the point $x$, and $\psi(x) = g(x)$, $\nabla\psi(x) = \nabla g(x) \neq 0$.*

**THEOREM 4.** *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary and $\psi \in W^{2,1}(\Omega)$. Then for $\mathfrak{H}^1$–almost all $y \in \psi(\overline{\Omega}) \subset \mathbb{R}$ the preimage $\psi^{-1}(y)$ is a finite disjoint family of $C^1$–curves $S_j$, $j = 1, 2, \dots, N(y)$. Each $S_j$ is either a cycle in $\Omega$ (i.e., $S_j \subset \Omega$ is homeomorphic to the unit circle $\mathbb{S}^1$) or it is a simple arc with endpoints on $\partial\Omega$ (in this case $S_j$ is transversal to $\partial\Omega$). Moreover, the tangent vector to each $S_j$ is an absolutely continuous function.*

# Lecture 7

In this lecture the detailed proof of the one-side maximum principle for the total head pressure (Bernoulli function) corresponding to a weak solution of the Euler equations will be presented.

Let $\Omega$ be a bounded domain with Lipschitz boundary. We say that the function $f \in W^1_s(\Omega)$ satisfies a *one-side maximum principle locally* in $\Omega$, if

$$\operatorname*{ess\,sup}_{x\in\Omega'} f(x) \le \operatorname*{ess\,sup}_{x\in\partial\Omega'} f(x) \tag{20}$$

holds for any strictly interior subdomain $\Omega'$ ($\overline{\Omega}' \subset \Omega$) with the boundary $\partial\Omega'$ not containing singleton connected components. Here negligible sets are the sets of 2–dimensional Lebesgue measure zero in the left *esssup*, and the sets of 1–dimensional Hausdorff measure zero in the right *esssup*.)

If (20) holds for any $\Omega' \subset \Omega$ (not necessary strictly interior) with the boundary $\partial\Omega'$ not containing singleton connected components, then we say that $f \in W^1_s(\Omega)$ satisfies a *one-side maximum principle* in $\Omega$ (in particular, we can take $\Omega' = \Omega$ in (20).

We shall prove the following

**THEOREM 5.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded multiply connected domain with Lipschitz boundary $\partial\Omega = \bigcup\limits_{i=1}^{N} S_i$. Let $\widehat{\mathbf{u}} \in W^1_2(\Omega)$ and $\widehat{p} \in W^1_s(\Omega)$, $s \in [1,2)$, satisfy Euler equations*

$$\begin{cases} (\widehat{\mathbf{u}}\cdot\nabla)\widehat{\mathbf{u}} + \nabla\widehat{p} &= 0, \\ \operatorname{div}\widehat{\mathbf{u}} &= 0 \end{cases}$$

*for almost all $x \in \Omega$ and $\int\limits_{S_i} \widehat{\mathbf{u}}\cdot\mathbf{n}\,dS = 0$, $i = 1,\dots,N$. Assume that there exists a sequence of functions $\{\Phi_\mu\}$ such that $\Phi_\mu \in W^1_{s,loc}(\Omega)$ and $\Phi_\mu \rightharpoonup \Phi$ in the space $W^1_{s,loc}(\Omega)$ for all $s \in [1,2)$. If all $\Phi_\mu$ satisfy the one–side maximum principle locally in $\Omega$, then $\Phi$ satisfies the one–side maximum principle in $\Omega$.*

The following two lemmas play an essential role in the proof of Theorem 5.

**LEMMA A.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $\widehat{\mathbf{u}}$, $p$ satisfy the conditions of Theorem 5. Assume that $K_i \subset \overline{\Omega}$ is a* *sequence of connected compact sets such that* $K_i \geq \delta > 0$ *and* $\psi|_{K_i} \equiv c_i = \text{const}$. *Suppose there exists a sequence* $x_i \in K_i$ *such that* $x_i \to x_0 \in K_0$, *where* $K_0 \subset \overline{\Omega}$ *is a connected compact set with* $\psi|_{K_0} = \text{const}$. *Then for any* $y_i \in K_i \setminus A_{\widehat{\mathbf{u}}}$ *and for any* $y_0 \in K_0 \setminus A_{\mathbf{w}}$ *the equality*

$$\lim_{i\to\infty} \Phi(y_i) = \Phi(y_0)$$

*holds.*

**LEMMA B.** *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded multiply connected domain with Lipschitz boundary. Let* $\widehat{\mathbf{u}} \in W^1_2(\Omega)$ *and* $p \in W^1_s(\Omega)$ *satisfy Euler equations for almost all* $x \in \Omega$ *and* $\int\limits_{S_i} \widehat{\mathbf{u}}\cdot\mathbf{n} dS = 0,\ i = 1,\dots,N$. *Assume that there exists a sequence of functions* $\{\Phi_\mu\}$ *such that* $\Phi_\mu \in W^1_{s,}(\Omega)$ *and* $\Phi_\mu \rightharpoonup \Phi$ *in* $W^1_{s,loc}(\Omega)$ *for all* $s \in [1,2)$. *Then for any subdomain* $\Omega' \subset \Omega$ *with* $X = X_{\Omega'} \neq \emptyset$ *the functions* $\Phi_\mu|_K$ *are continuous on almost all admissible cycles* $K$ *and the sequence* $\{\Phi_\mu|_K\}$ *converges to* $\Phi|_K$ *uniformly:*

$$\Phi_\mu|_K \rightrightarrows \Phi|_K.$$

In particular, it follows from Theorem 5 that if $\widehat{\mathbf{u}}\big|_{\partial\Omega} = 0$ (in the sense of traces), then

$$\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) \leq \operatorname*{ess\,sup}_{x\in\partial\Omega} \Phi(x) = \max\{\widehat{p}_1, \widehat{p}_2, \dots, \widehat{p}_N\},$$

where $\widehat{p}(x)|_{S_i} = \widehat{p}_i = \text{const}$.

Note that some version of a local weak one–side maximum principle was proved by Ch. Amick. However, his result was not enough for our purposes.

# Lecture 8

The nonhomogeneous boundary value problem for the steady Navier–Stokes equations will be studied in a two–dimensional exterior domain $\Omega$. It is assumed that the domain $\Omega$ and the boundary value $\mathbf{a}$ are symmetric with respect to the $x_1$-axis. The existence of a solution to this problem will be proved for arbitrary values of the fluxes $F_i$ of the boundary value $\mathbf{a}$.

Let $\Omega \subset \mathbb{R}^2$ be an exterior domain

$$\Omega = \mathbb{R}^2 \setminus \bigcup_{j=1}^{N} \Omega_j, \ N \geq 1,$$

where $\Omega_j$ are bounded domains with Lipschitz boundaries $S_j$,

$$\Omega_j \cap \Omega_j = \emptyset, j \neq i.$$

Suppose that $\Omega$ is symmetric with respect to $x_1$–axis, i.e.,

$$(x_1, x_2) \in \Omega \Leftrightarrow (x_1, -x_2) \in \Omega,$$

and suppose that all domains $\Omega_j$ intersect this axis.

We say that the vector function $\mathbf{u}(x_1, x_2)$ is symmetric, if $u_1$ is an even function of $x_2$ and $u_2$ is odd function of $x_2$ and $u_2$:

$$u_1(x_1, x_2) = u_1(x_1, -x_2), \quad u_2(x_1, x_2) = -u_2(x_1, -x_2).$$

Consider in $\Omega$ the boundary value problem for the steady–state plane Navier–Stokes equations

$$\begin{cases} -\nu\Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{v} &= 0 \quad \text{in } \Omega, \\ \mathbf{v} &= \mathbf{a} \quad \text{on } \partial\Omega, \end{cases}$$

where

$$\int\limits_{\partial\Omega} \mathbf{a}(x)\cdot\mathbf{n}(x)\, dS = F \neq 0.$$

Here $\mathbf{n}$ is the outward (with respect to $\Omega$) unit normal to $\partial\Omega$. Notice that

$$\int\limits_{\partial\Omega} \mathbf{a}(x)\cdot\mathbf{n}(x)\, dS = \sum_{j=1}^{M} \int\limits_{S_j} \mathbf{a}(x)\cdot\mathbf{n}(x)\, dS = \sum_{j=1}^{M} F_j$$

and that we do not assume that the fluxes $F_j$ or the total flux $F$ are small.

We shall prove the existence of at least one weak symmetric solution to the above problem. This solution has finite Dirichlet integral $\int\limits_{\Omega} |\nabla \mathbf{v}|^2 dx < \infty$. It is known that $\mathbf{v}$ tends to some constant at infinity

$$\lim_{|x|\to\infty} \mathbf{v} = \mathbf{v}_\infty = const.$$

In general, we cannot prescribe this constant. However, we shall prove that in the case of domains and boundary values that are symmetric with respect to two coordinate axis the solution tend at infinity to zero:

$$\lim_{|x|\to\infty} \mathbf{v} = 0.$$

The weak solution $\mathbf{v}$ of the problem in the exterior domain will be found as a limit of a sequence of solutions $\mathbf{v}_R$ in bounded domains $\Omega_R = \Omega \cap \{x : |x| < R\}$. In order to prove a uniform with respect to $R$ a priori estimate of solutions $\mathbf{v}_R$, we construct a special extension $\mathbf{B}(x, \varepsilon)$ of the boundary value $\mathbf{a}$ which satisfies the Leray–Hopf's inequality. The essential role in proving the a priory estimate plays also the following new inequality:

**LEMMA C.** *Suppose that* $v|_{\partial\Omega} = 0$ *and* $\int\limits_{\Omega} |\nabla \mathbf{v}|^2 dx < \infty$. *Let* $\kappa > 0$, $\alpha \in (1/2, 1)$. *Then the inequality*

$$\int\limits_{\mathbb{R}\setminus(-a,a)} \int\limits_{0}^{\kappa|x_1|^\alpha} \frac{|v(x_1,x_2)|^2}{|x|^2} dx_1 dx_2 \le c \int\limits_{\Omega} |\nabla v(x)|^2 dx.$$

*holds. Here* $a$ *is such that*

$$\bigcup_{j=1}^{N} \Omega_j \subset \{x : |x| < a\}.$$

# Lecture 9

In this lecture we discuss the generalizations of results obtained in the two-dimensional case to the three-dimensional bounded axially symmetric domains.

Let $\Omega = \Omega_0 \setminus \bigcup_{j=1}^{N} \Omega_j$ be a bounded domain in $\mathbb{R}^3$ with multiply connected Lipschitz boundary $\partial\Omega$ consisting of $N+1$ disjoint components $\partial\Omega_j = S_j$: $\partial\Omega = S_0 \cup \ldots \cup S_N$, $S_i \cap S_j = \emptyset$, $i \neq j$. Consider in $\Omega$ the stationary Navier–Stokes system with nonhomogeneous boundary conditions

$$\begin{cases} -\nu\Delta \mathbf{v} + \big(\mathbf{v}\cdot\nabla\big)\mathbf{v} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{a} & \text{on } \partial\Omega, \end{cases}$$

where

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\, dS = \sum_{j=0}^{N} \int\limits_{S_j} \mathbf{a}\cdot\mathbf{n}\, dS = \sum_{j=0}^{N} F_i = 0.$$

We shall study the problem in the axial symmetric case. Let $O_{x_1}, O_{x_2}, O_{x_3}$ be coordinate axis in $\mathbb{R}^3$, $(\theta, r, z)$ be cylindrical coordinates and $v_\theta, v_r, v_z$ be the projections of the vector $\mathbf{v}$ on the axes $\theta, r, z$. A vector-valued function $\mathbf{h} = (h_\theta, h_r, h_z)$ is called *axially symmetric* if $h_\theta$, $h_r$ and $h_z$ do not depend on $\theta$, and $\mathbf{h} = (h_\theta, h_r, h_z)$ is called *axially symmetric without rotation* if $h_\theta = 0$ while $h_r$ and $h_z$ do not depend on $\theta$. We will use the following symmetry assumptions.

(SO) $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary and $O_{x_3}$ is the axis of symmetry of the domain $\Omega$.

(AS) The assumptions (SO) are fulfilled and the boundary value $\mathbf{a} \in W_2^{1/2}(\partial\Omega)$ is axially symmetric.

(ASwR) The assumptions (SO) are fulfilled and the boundary value $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ is axially symmetric without rotation.

Assume that

$$S_j \cap O_{x_3} \neq \emptyset, \quad j = 0, \ldots, M, \quad S_j \cap O_{x_3} = \emptyset, \quad j = M+1, \ldots, N.$$

We will prove the existence theorem for the solution if one of the following two additional conditions is fulfilled:

$$M = N-1, \qquad F_N \geq 0, \tag{A}$$

or

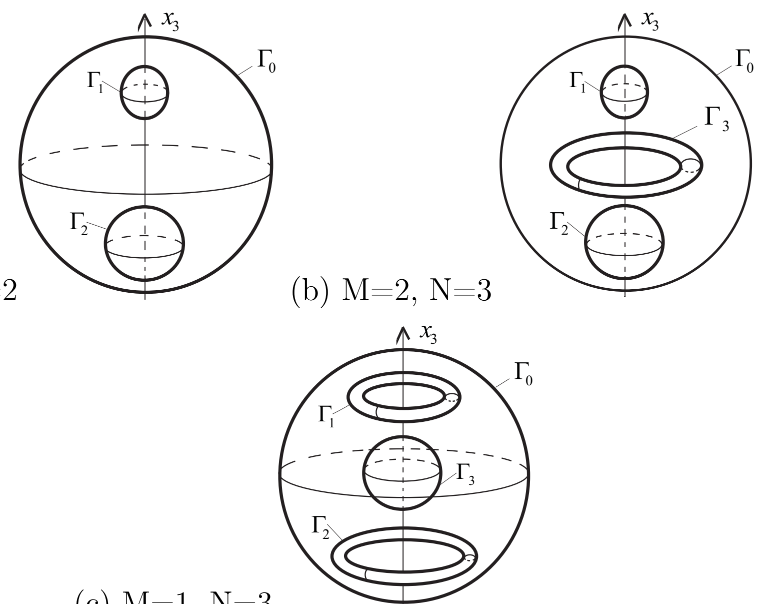
$$|F_j| < \delta, \quad j = M+1, \dots, N, \tag{A}$$

where $\delta$ is sufficiently small. In particular, (B) includes the case $N = M$, i.e, when each component of the boundary intersects the axis of symmetry. In both cases (A) and (A) the fluxes $F_j, j = 0, 1, \dots, M$, are arbitrary.

The main result reads as follows.

**THEOREM 6.** *Let the conditions* (AS) *be fulfilled. Suppose that fluxes $F_j$ satisfy the necessary solvability condition and also that one of the conditions (A) or (B) holds. Then the Navier–Stokes problem admits at least one weak axially symmetric solution.*

*If, in addition, the conditions* (ASwR) *are fulfilled, then there exists at least one weak axially symmetric solution without rotation.*



(a) M=N=2 (b) M=2, N=3 (c) M=1, N=3

Figure 1: Domain $\Omega$

# Lecture 10

In the last lecture we consider the stationary Navier–Stokes system with nonhomogeneous boundary conditions in a class of domains $\Omega$ having "paraboloidal" outlets to infinity.

We assume that the boundary $\partial\Omega$ is multiply connected and consists of $M$ infinite connected components $S_m$ which form the outer boundary $S = \bigcup\limits_{m=1}^{M} S_m$, and $I$ compact connected components $\Gamma_i$ forming the inner boundary $\Gamma = \bigcup\limits_{i=1}^{I} \Gamma_i$. Concerning the boundary value $\mathbf{a}$ we assume that it has compact support and we suppose that the fluxes $\mathbb{F}_i^{(inn)}$ of $\mathbf{a}$ over the connected components $\Gamma_i$ of the inner boundary are sufficiently small. We do not have any restrictions on fluxes $\mathfrak{F}_m^{(out)}$ of $\mathbf{a}$ over the components $S_m$ of the infinite outer boundary. Under these conditions we prove the existence of at least one weak solution of the Navier–Stokes problem which has prescribed fluxes $F_j$ over the cross-sections of outlets to infinity. Of course the necessary compatibility condition

$$\sum_{i=1}^{I} \mathbb{F}_i^{(inn)} + \sum_{m=1}^{M} \mathfrak{F}_m^{(out)} + \sum_{j=1}^{J} F_j = 0$$

should be valid (i.e., the total flux should be equal to zero).

The solution can have finite or infinite Dirichlet integral depending on geometrical properties of the outlets (the solution has finite Dirichlet integral, if the outlets $D_j,\ j = 1, \dots, J$, are sufficiently "wide"). We give a constructive proof of the existence of the solution based on special construction of the extension $\mathbf{A}$ of the boundary value $\mathbf{a}$ into the domain $\Omega$. This extension is constructed as a sum

$$\mathbf{A} = \mathbf{B}^{(inn)} + \sum_{m=1}^{M} \mathbf{B}_m^{(out)} + \mathbf{B}^{(flux)},$$

where $\mathbf{B}^{(inn)}$ extends the boundary value $\mathbf{a}$ from the inner boundary $\Gamma$, $\mathbf{B}_m^{(out)}$ extend $\mathbf{a}$ from the connected component $S_m$ of the outer boundary, and $\mathbf{B}^{(flux)}$ has zero boundary value and removes the fluxes $F_j$. The vector fields $\mathbf{B}_m^{(out)}$ and $\mathbf{B}^{(flux)}$ are constructed to satisfy the Leray–Hopf's inequality which allows to obtain a priori estimates of the solution for arbitrary large fluxes $\mathfrak{F}_m^{(out)}$ and $F_j$.

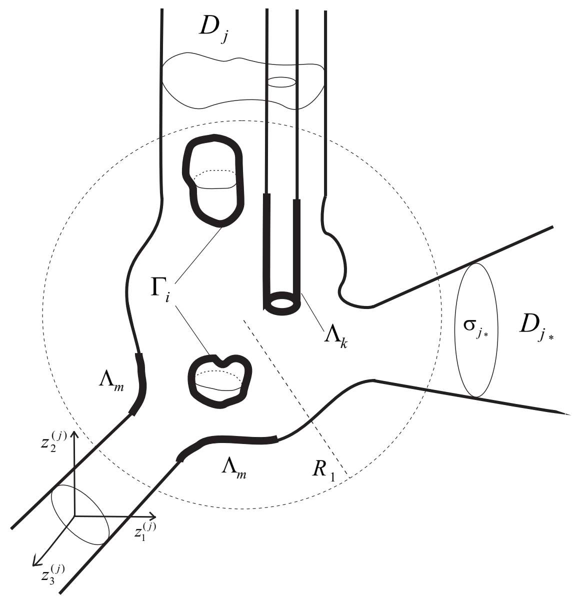



Figure 2: Domain $\Omega$

We mention that, in general, the Leray–Hopf's inequality cannot be true for the vector field $\mathbf{B}^{(inn)}$ (because of the counterexample of Takashita). Therefore, we have to suppose that the fluxes $\mathbb{F}_i^{inn}$ of $\mathbf{a}$ over the compact components of the inner boundary are "sufficiently small". After the extension $\mathbf{A}$ with above properties is constructed, the proof of the existence of a weak solution of the Navier–Stokes problem is just the same as in the case of homogeneous boundary data.